A Momentum Construction for Circle-Invariant Kähler Metrics*

Andrew D. Hwang†
University of Toronto

Michael A. Singer
University of Edinburgh

Contents

1 Introduction

The subject of this paper is the explicit construction of complete Kähler metrics with prescribed—usually constant—scalar curvature. The technique, hereafter referred to as the *momentum construction*, is a combination of two main ideas. The first, which goes back at least to the work of Calabi [4], is that of constructing Kähler forms from Kähler potentials that are essentially functions of one real variable. The prototypical example is the ansatz $\omega = \sqrt{-1}\partial\bar{\partial} f(t)$, where

*AMS Subject Classification: 32L05, 53C21 (Primary); 53C55, 32L07 (Secondary).
†Supported in part by JSPS Fellowship #P-94016 and an NSERC Canada Research Grant

$t = (1/2)\log(|z_1|^2 + \cdots + |z_n|^2)$, the z_j are standard complex coordinates on $\mathbf{C}^n$, and f is a smooth function of one real variable. As is well-known, if $f(t) = t$, then ω descends to the Fubini-Study metric on $\mathbf{P}^{n-1}$, while if $f(t) = e^t$, ω extends to the standard flat Kähler form on $\mathbf{C}^n$.

The general setting for a construction of this type is the total space of an Hermitian holomorphic line bundle $p : (L, h) \to (M, \omega_M)$ over a Kähler manifold. Let t be the logarithm of the fibrewise norm function defined by h and consider the *Calabi ansatz*

$$\omega = p^*\omega_M + 2\sqrt{-1}\partial\bar{\partial}f(t), \tag{1.1}$$

which associates a Kähler form to (suitably convex) functions of one real variable.

The second idea comes from symplectic geometry and involves a change of variables $f(t)$ to $\varphi(\tau)$ closely related to the Legendre transform. For each f, (1.1) is invariant under the circle-action that rotates the fibres of L. Let X be the generator of this action, normalized so that $\exp(2\pi X) = 1$. Denote by τ the corresponding *moment map* determined (up to an additive constant) by

$$i_X\omega = -d\tau. \tag{1.2}$$

(In fact $\tau = f'(t)$, see Section 2.1.) When ω comes from the Calabi ansatz, the function $\|X\|^2_\omega$ is constant on the level sets of τ, so there is a function $\varphi : I \to (0, \infty)$—to be called the *momentum profile* of ω—such that

$$\varphi(\tau) = \|X\|^2_\omega. \tag{1.3}$$

The crucial point is that ω can be reconstructed explicitly from its profile; indeed t and τ are related by the Legendre transform, and the Legendre dual F of f satisfies $F'' = 1/\varphi$.

The description of ω in terms of φ has many advantages. At one level, this is to be expected: (1.3) shows that φ is a canonical geometric quantity, while f determines ω only through its second derivative. In particular, the only condition that is needed for φ to determine a Kähler metric is that it be positive on (the interior of) I, cf. (1.3). Positivity of (1.1), by contrast, corresponds to two conditions on the derivatives of f. Further, the geometry of the metric—for example its completeness or extendability properties near the fixed-point set of the S^1-action—are easily read off from the behaviour of φ near the endpoints of I. However, the decisive and most remarkable advantage, as far as the problem of prescribed scalar curvature is concerned, is that the scalar curvature of ω is given by a *linear* second-order differential expression in $\varphi(\tau)$, in contrast to the fully nonlinear fourth-order function that arises in the $f(t)$-description. Consequently, the profiles that give rise to metrics of constant scalar curvature are explicit rational functions of τ.

This dramatic simplification is already apparent in the basic example of S^1-invariant metrics on S^1-invariant domains of $\mathbf{P}^1$. In this case, the scalar curvature is given by $-\varphi''(\tau)/2$. For the isometric S^1-action on the unit sphere in $\mathbf{R}^3$ by rotations about the z-axis, the coordinate z itself is the moment map τ, and $\varphi(\tau) = 1 - \tau^2$. It is immediately *calculated* that this metric has constant scalar curvature. Generally, it is a pleasant and instructive exercise to classify circle-invariant metrics with constant scalar curvature on subsets of S^2 from this point of view, and to compare the simplicity of the results with other approaches to this problem. A sketch is provided in Section 2.3 (Geometry of fibre metrics; see also Table 2.2, page 18).

This paper is organized as follows. The remainder of the introduction summarizes our main results. Section 2 gives a self-contained account of the momentum construction and Section 3

applies the method to give general existence theorems for complete Kähler metrics of constant scalar curvature. Finally Section 4 is devoted to a discussion of three topics: the scope and limitations of the momentum construction; examples of line bundles (L, h) to which the results of Section 3 apply; and an account of the related literature. While postponing until that section a careful explanation of the ways in which our work builds on and extends that of previous authors, let us pause here to acknowledge the most important sources of inspiration for the present work: the papers of Calabi [4], Koiso–Sakane [16], LeBrun [19], and Pedersen–Poon [27].

Acknowledgements The first-named author was supported in part by a JSPS Postdoctoral Research Fellowship during the early stages of this work, and by an NSERC Canada Individual Research Fellowship. He is especially grateful to Professors Y. Sakane and T. Mabuchi, and to Osaka University, for kind and extensive hospitality, and to Professor J. Bland, of the University of Toronto, for many patient and helpful discussions. The second-named author is an EPSRC advanced fellow. He is also grateful to the University of Toronto for its hospitality while some of this work was being carried out.

Results about the momentum construction

In the rest of the paper, $p : (L, h) \to (M, \omega_M)$ is an Hermitian holomorphic line bundle, with curvature form $\gamma = -\sqrt{-1}\partial\bar{\partial} \log h$, over a Kähler manifold of complex dimension m. A *compatible momentum interval* is an interval $I \subset \mathbf{R}$, such that the closed $(1,1)$-form $\omega_M(\tau) := \omega_M - \tau\gamma$ is positive for every $\tau \in I$. The associated Kähler metric is denoted $g_M(\tau)$.

Definition 1.1 *Horizontal data* $\{p : (L, h) \to (M, g_M),\ I\}$ consists of an Hermitian holomorphic line bundle over a Kähler manifold, together with a compatible momentum interval. The whole assemblage is often denoted $\{p, I\}$ for brevity. A *momentum profile* is a function φ that is smooth on the closure of I and positive on the interior of I.

The *completion* of a line bundle L is the $\mathbf{P}^1$ bundle $\widehat{L} = \mathbf{P}(\mathcal{O} \oplus L)$, containing L as a Zariski-open subset and obtained by adding a copy of M 'at ∞.' Let $r : \widehat{L} \to [0, \infty]$ denote the continuous extension of the square of the Hermitian norm function. All the metrics constructed in this paper live on subsets of $\widehat{L}$ obtained by restricting r to an interval (or on manifolds obtained by partially collapsing the zero and/or infinity sections).

Definition 1.2 Let $\mathrm{J} \subset [0, \infty]$ be an open interval. The corresponding *invariant subbundle* $L' \subset \widehat{L}$ is the S^1-invariant domain $r^{-1}(\mathrm{J})$.

Different choices of J yield six distinct complex-analytic fibre types: $\mathrm{J} = [0, \infty]$ (the projective line); $\mathrm{J} = [0, \infty)$ (the complex line); $\mathrm{J} = (0, \infty)$ (the punctured line); $\mathrm{J} = [0, 1)$ (the disk); $\mathrm{J} = (0, 1)$ (the punctured disk); and $\mathrm{J} = (e^{-l}, e^{l})$ (annuli). In the last three cases, homotheties have been used to reduce J to a standard form. The invariant subbundles corresponding to the first five cases will be denoted $\widehat{L}$, L, $L^\times$, $\Delta(L)$ and $\Delta^\times(L)$ respectively. Annulus-subbundles will not play a major role in what follows.

Definition 1.3 Let $L' \subset \widehat{L}$ be an invariant subbundle. A *bundle-adapted metric* on L' is a Kähler metric g whose Kähler form ω arises from the Calabi ansatz (1.1).

The heart of the momentum construction is the fact, implicitly due to Calabi and Koiso-Sakane, that if horizontal data are given, then each momentum profile determines a unique isometry class of bundle-adapted Kähler metric enjoying the geometric properties of equations (1.1)–(1.3):

Proposition 1.4 *Let horizontal data $\{p, I\}$ and a momentum profile φ be given. Then there exists an invariant subbundle $L' \subset \widehat{L}$, unique up to homothety, and a bundle-adapted Kähler metric g_φ on L', unique up to isometry, with the following properties:*

(i) *The Kähler form ω_φ of g_φ arises from the Calabi ansatz;*

(ii) *The image of the moment map τ is I;*

(iii) *The normalized generator X of the circle action satisfies $g_\varphi(X, X) = \varphi(\tau)$.*

It is natural to ask how the geometric properties, especially completeness and the scalar curvature σ_φ, of g_φ are encoded by φ. As will be seen in Section 2.3 below, completeness of g_φ is encoded by the boundary behaviour of φ, in fact by the 2-jet at the endpoints of I. As claimed above, the scalar curvature is linear in φ:

Theorem A *Let horizontal data be given. For each $\tau \in I$, let $\sigma_M(\tau)$ denote the scalar curvature of $g_M(\tau)$, and define $Q : I \times M \to \mathbf{R}$ by $Q(\tau) = \omega_M(\tau)^m/\omega_M^m$. Then the scalar curvature of g_φ is given by*

$$\sigma_\varphi = \sigma_M(\tau) - \frac{1}{2Q}\frac{\partial^2}{\partial\tau^2}(Q\varphi)(\tau). \tag{1.4}$$

To interpret (1.4), regard the S^1-invariant function σ_φ on L' as a function on $I \times M$ by factoring through the S^1-action.

The improvement of Theorem A over earlier results is that no curvature hypotheses are imposed on the horizontal data. Of course, it is too much to expect that for arbitrary horizontal data, σ_φ can be made constant by an appropriate choice of profile (a single function of one variable). Natural *sufficient* curvature hypotheses are to assume the two terms in (1.4) separately depend only on τ for every profile:

Definition 1.5 Horizontal data are said to be *σ-constant* if

(i) The curvature endomorphism $\mathrm{B} = \omega_M^{-1}\gamma$ has constant eigenvalues on M;

(ii) The metric $g_M(\tau)$ has constant scalar curvature for each $\tau \in I$.

The simplest examples of σ-constant horizontal data are pluricanonical bundles over Einstein-Kähler manifolds. More generally, if g_M has constant scalar curvature, and $\gamma(L, h)$ is a multiple of ω_M, then the data $p : (L, h) \to (M, \omega_M)$ are σ-constant. More specific examples appear in Section 4.

The conclusion of Theorem A is due to Guan [10] and Hwang [12] under stronger curvature hypotheses ("ρ-constancy," see Section 4.5). The corresponding statement for σ-constant data is used repeatedly in the sequel:

Corollary A.1 *Let the horizontal data $\{p : L \to M,\ I\}$ be σ-constant. Then $\sigma_M(\tau) = P/2Q$, where Q is as in Theorem A, P is a polynomial in τ, and for each momentum profile φ,*

$$\sigma_\varphi = \frac{1}{2Q}\Big(P - (Q\varphi)''\Big)(\tau). \tag{1.5}$$

Metrics of constant scalar curvature on line bundles

The next two theorems are generalizations of the work of various authors. Detailed attribution is given in Section 4.5.

Theorem B *Let $I = [0, \infty)$, and let $\{p, I\}$ be σ-constant horizontal data with $\gamma \le 0$, $\gamma \ne 0$, and with g_M complete. Then there exists a real number c_0 with the following property: For every $c < c_0$, and for at most finitely many $c > c_0$, the disk bundle $\Delta(L)$ carries a complete Kähler metric g_c, of scalar curvature c, whose restriction to the zero section is g_M. When $c = c_0$, the analogous conclusion holds, but the metric lives on the total space of L. For $c \le 0$, the metric g_c is Einstein iff $\frac{c}{m+1}\omega_M = \rho_M + \gamma$.*

An analogue of Theorem B holds for punctured disk bundles. Of course, the metrics are not obtained from the metrics in Theorem B by restriction, since removing the zero section leaves an incomplete metric.

Theorem C *Let $I = (0, \infty)$, and let $\{p, I\}$ be σ-constant horizontal data with $\gamma \le 0$, $\gamma \ne 0$, and with g_M complete. Then there exists a real number $c_0^\times$ with the following property: For every $c < c_0^\times$, and for at most finitely many $c > c_0^\times$, $\Delta^\times(L)$ carries a complete Kähler metric $g_c^\times$, of scalar curvature c, whose symplectic reduction at $\tau = 1$ is $g_M - \gamma$. When $c = c_0^\times$, the analogous conclusion holds, but the metric lives on the total space of $L^\times$. For $c < 0$, the metric g_c is Einstein iff $\frac{c}{m+1}\omega_M = \rho_M$.*

Remark 1.6 There is some redundancy in these statements; the condition $I = (0, \infty)$ implies that γ is non-positive, for $\omega_M(\tau)$ is supposed to be positive for all $\tau \in I$. On the other hand, it is natural to exclude the case $\gamma = 0$, since bundle-adapted metrics on flat bundles are local product metrics, and can thus be understood in an elementary fashion. □

Remark 1.7 The choice $I = [0, \infty)$ or $(0, \infty)$ is a normalization of the data which results in no loss of generality, see Lemma 4.1 below. □

Remark 1.8 The constants c_0 and $c_0^\times$ are roots of polynomials in one variable and can be estimated in terms of the horizontal data. In good cases they can be found exactly. In particular, if γ is negative-definite then $c_0 = 0$ and one has the pleasant conclusion that for every $c < 0$, $\Delta(L)$ admits a complete Kähler metric with scalar curvature c, while L admits a complete scalar-flat Kähler metric. □

Theorem B contains many previously-known results (see Section 4), but even when M is a complex curve, an interesting new result is obtained:

Corollary B.1 *Let $M = \mathbf{C}$, $\omega_M = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$ the standard flat Kähler form, $(L, h) \to M$ the trivial line bundle equipped with an Hermitian metric of curvature $\gamma = -2\pi k\, \omega_M$, ($k$ a positive constant). Then the total space of L—biholomorphic to $\mathbf{C}^2$—admits a complete, scalar-flat Kähler metric that is not Ricci-flat, and whose fibre metric is asymptotically cylindrical.*

The importance of this example is that scalar-flat Kähler metrics on complex surfaces are anti-self-dual (in the sense of 4-dimensional conformal geometry), and are thus of independent interest.

The following Einstein-Kähler metrics arising from Theorem B are due to Calabi [4] when the base is Einstein-Kähler (i.e. when $n = 1$ in Corollary B.2 below). Moreover, non-homothetic metrics are obtained in Corollary B.2 by scaling the ω_{M_i} separately. In Corollary B.3, further metrics arise by tensoring with a flat line bundle when the base is not simply-connected.

Corollary B.2 *For $1 \leq i \leq n$, let $p_i : K_i \to M_i$ be the canonical bundle of a compact Einstein-Kähler manifold of positive curvature. Then there is a complete, Ricci-flat Kähler metric on the total space of $\bigotimes_i p_i^* K_i$.*

Corollary B.3 *For $1 \leq i \leq n$, let $p_i : K_i \to M_i$ be a compact Einstein-Kähler manifold of positive curvature. Then for all $\ell > 1$, there is a complete Einstein-Kähler metric of negative curvature on the disk subbundle of $(\bigotimes_i p_i^* K_i)^\ell$. If M_i are complete, negative Einstein-Kähler manifolds, then the disk subbundle in $(\bigotimes_i p_i^* K_i^{-1})^\ell$ admits a complete, negative Einstein-Kähler metric.*

Theorem C yields some new Einstein-Kähler metrics on punctured disk bundles. The fibre metric has one end of finite area and one end of infinite area.

Corollary C.1 *Let $p : L \to (M, g_M)$ be a holomorphic line bundle over a complete, negative Einstein-Kähler manifold, and assume $c_1(L) \in H^{1,1}(M, \mathbf{Z})$ is represented by a form γ whose eigenvalues with respect to ω_M are constant on M. Then $\Delta^\times(L)$ admits a complete, negative Einstein-Kähler metric.*

Metrics of constant scalar curvature on vector bundles

The Calabi ansatz applies, in modified form, to vector bundles $E \to D$ of rank $n > 1$; Calabi's construction of a complete, Ricci-flat metric on the cotangent bundle of a compact, rank-one Hermitian symmetric space [4] is the prototypical example. In seeking generalizations of this example, one adopts the following strategy. Take $M = \mathbf{P}(E)$, $L = \tau_E$ the tautological line bundle over $\mathbf{P}(E)$. Since the total space of L is the blow-up along the zero-section of E, a metric on E is the same thing as a suitably degenerate metric on τ_E. The momentum construction is easily modified to produce such partially degenerate metrics on L:

Theorem D *Let $(E, h) \to (D, g_D)$ be an Hermitian holomorphic vector bundle of rank n over a complete Kähler manifold of dimension d, and let $p : (\tau_E, h) \to M = \mathbf{P}(E)$ be the tautological bundle with the induced Hermitian structure. Let γ denote the curvature form of (τ_E, h). Assume $\omega_M(\tau) := \omega_D - \tau\gamma$ is a Kähler form on M for all $\tau > 0$, and that $p : (\tau_E, h) \to (M, g_M(\tau_0))$*

is σ-constant for some $\tau_0 > 0$. Then there is a c_0 such that for every $c < c_0$, and for at most finitely many $c > c_0$, the momentum construction determines a complete Kähler metric with scalar curvature c on the disk subbundle $\Delta \subset E$. When $c \leq 0$, this metric is Einstein if and only if $\frac{c}{n+d}\omega_D = \rho_M + n\gamma$. Furthermore, there is a complete Kähler metric on E with scalar curvature c_0.

Unfortunately, σ-constancy of the data $\tau_E \to \mathbf{P}(E)$ is a strong assumption. It is not even sufficient for E to be a homogeneous vector bundle, as the example $E = \mathcal{O} \oplus \mathcal{O}(k) \to \mathbf{P}^1$ $(k < 0)$ shows. In this case $\mathbf{P}(E)$ is a Hirzebruch surface, which does not admit a Kähler metric of constant scalar curvature, so condition (ii) in the definition of σ-constancy is never satisfied.

Each of the next three corollaries contains tractable hypotheses that guarantee the applicability of Theorem D. In Corollary D.1, it is assumed that both E and $\mathbf{P}(E)$ are homogeneous. The pseudoconvexity hypothesis guarantees a metric of non-positive curvature on the tautological bundle. The hypotheses of Corollary D.2 imply that $\mathbf{P}(E)$ is a product and that τ_E is a tensor product of line bundles pulled back from the factors. Corollary D.3, which rests on the result of Narasimhan-Seshadri to the effect that a stable bundle over a curve is projectively flat, gives families of examples depending on continuous parameters in the event that the base of E is one-dimensional. As above, the restriction of the advertized metric to the zero section is the base metric g_D. In particular, varying the scalar curvature does not vary the metric through homotheties.

Corollary D.1 *Let $E \to D$ be a homogeneous vector bundle of rank $n > 1$ over a compact, homogeneous Kählerian manifold. Assume further that the projective bundle $M = \mathbf{P}(E)$ is homogeneous, and that the zero section of E has a pseudoconvex tubular neighborhood. Endow the base with a homogeneous Kähler metric g_D, and the bundle with a homogeneous Hermitian metric h. Then for every $c < 0$, there is a complete Kähler metric g_c of scalar curvature c on the disk bundle Δ, and there is a complete, scalar-flat metric on the total space of E.*

Corollary D.2 *Suppose (D, g_D) is complete and that $(\Lambda, h) \to (D, g_D)$ is σ-constant, with curvature $\gamma < 0$. Then the total space of $E = \Lambda \otimes \mathbf{C}^n = \Lambda^{\oplus n}$ admits a complete, scalar-flat metric, and for every $c < 0$, the disk bundle Δ (taken with respect to the natural Hermitian structure) admits a complete Kähler metric of scalar curvature c.*

Corollary D.3 *Let $p : E \to C$ be a stable vector bundle of rank n and degree $k \leq 0$ over a compact Riemann surface C, of genus $g \geq 2$ and endowed with the unit-area metric of constant Gaussian curvature. Then the total space of E admits a complete, scalar-flat Kähler metric, and there exists an Hermitian structure h such that for every $c < 0$, the disk subbundle of (E, h) carries a complete Kähler metric of scalar curvature c. Finally, if $k \leq 2 - 2g$, then there is a complete Einstein-Kähler metric with scalar curvature $c = 2\pi(n+1)(2g - 2 + k)$, on E if $c = 0$ and on $\Delta \subset E$ if $c < 0$.*

Limitations of the Calabi ansatz

If horizontal data are σ-constant, then by (i) of Definition 1.5, $Q(\tau)$ is constant for all $\tau \in I$, so the second term in (1.4) depends only on τ for every profile φ, while (ii) says exactly that $\sigma_M(\tau)$ is

constant for all τ. Conversely, it is not difficult to show that if, *for every* φ, σ_φ depends only upon τ then the horizontal data are σ-constant. We believe the same conclusion follows merely if *there exists* a profile inducing a metric of constant scalar curvature, but are at present able to prove only a partial result in this direction:

Theorem E *Let $\{p, I\}$ be horizontal data with $I = [0, \varepsilon)$ for some $\varepsilon > 0$, and assume there exist real-analytic functions σ_1 and σ_2 on I, and distinct profiles φ_1 and φ_2, with $\varphi_1(0) = \varphi_2(0)$ and inducing metrics of scalar curvature $\sigma_1(\tau)$ and $\sigma_2(\tau)$, respectively. Then the horizontal data are σ-constant.*

Theorem E is proved by considering the difference $\varphi_1 - \varphi_2$, and is of course purely local. A simple but suggestive corollary is that on horizontal data that are not σ-constant, there is at most one metric of constant scalar curvature arising from the Calabi ansatz. In the statement below, it is not necessary to assume $c_1 \neq c_2$; if $c_1 = c_2$, then φ_1 and φ_2 are distinct iff $\varphi_1'(0) \neq \varphi_2'(0)$.

Corollary E.1 *Let $\{p, I\}$ be horizontal data as above, and assume there exist distinct profiles φ_1 and φ_2, equal at 0 and inducing metrics of scalar curvature c_1 and c_2, respectively. Then the horizontal data are σ-constant.*

There does not seem to be an easy way to prove σ-constancy by assuming existence of *one* profile, without additional hypotheses. Further partial results and a conjectural strengthening of Theorem E are described in Section 4.

2 Bundle-Adapted Metrics

This section gives a full and self-contained account of the momentum construction. To facilitate the presentation, a detailed account of the credit due to earlier authors is postponed to Section 4.5. We begin with a short summary of the most important formulae of the momentum construction. Proofs and further comments come in Section 2.3.

2.1 The Momentum Construction

Terminology is as in Definitions 1.1–1.3. The following notational conventions are used systematically: A Kähler metric is denoted g. Its Kähler form, Ricci form, scalar curvature, and Laplace operator (acting on functions) are denoted ω, ρ, σ, and $\Box$ respectively. Subscripts are used descriptively when several metrics are under consideration.

Fix $b \leq \infty$. Let $I \subset \mathbf{R}$ be an interval with interior $(0, b)$, and pick $\tau_0 \in (0, b)$. For each profile φ, define

$$t_1 = \lim_{\tau \to 0^+} \int_{\tau_0}^{\tau} \frac{dx}{\varphi(x)}, \quad t_2 = \lim_{\tau \to b^-} \int_{\tau_0}^{\tau} \frac{dx}{\varphi(x)}, \tag{2.1}$$

noting that $-\infty \leq t_1 < t_2 \leq \infty$. Introduce functions μ, s, and f of t by

$$t = \int_{\tau_0}^{\mu(t)} \frac{dx}{\varphi(x)}, \qquad s(t) = \int_{\tau_0}^{\mu(t)} \frac{dx}{\sqrt{\varphi(x)}}, \qquad f(t) = \int_{\tau_0}^{\mu(t)} \frac{x\,dx}{\varphi(x)}. \tag{2.2}$$

Then the equations

$$\mu' = \varphi \circ \mu, \qquad s' = \sqrt{\varphi \circ \mu}, \qquad f' = \mu \tag{2.3}$$

hold on (t_1, t_2).

Suppose now that $\{p : (L, h) \to (M, g_M), I\}$ are horizontal data, and let $t : L^\times \to \mathbf{R}$ be the logarithm of the norm-function. The functions μ, s, and f of (2.2) induce respective functions $\tau := \mu(t)$, $s(t)$, and $f(t)$ on $L' = \{z \in L^\times : t_1 < t(z) < t_2\}$, and determine a closed $(1,1)$-form

$$\omega_\varphi = p^*\omega_M + dd^c f(t) = p^*\omega_M + 2\sqrt{-1}\partial\bar{\partial} f(t). \tag{2.4}$$

It transpires that ω_φ is positive, i.e. is a Kähler form. The associated bundle-adapted metric will be denoted g_φ, and the mapping $\varphi \longmapsto \omega_\varphi$ will be called the *momentum construction*.

The fibre metric

If z^0 is a linear coordinate on a fibre of L then the restriction of g_φ to the fibre is given by

$$g_{\text{fibre}} = \varphi(\tau) \left| \frac{dz^0}{z^0} \right|^2. \tag{2.5}$$

Thus $\varphi(\tau)$ is interpreted as the conformal factor relating g_{fibre} to the flat metric on the cylinder. The function s of (2.2) is the geodesic distance for g_{fibre}, while $2\pi(\mu(b) - \mu(a))$ is the area of the subset of the fibre given by $a \le t \le b$.

S^1-invariant functions on L

Fix a bundle-adapted metric on L'. The functions t and $\tau = \mu(t)$ have the same level sets on L', but while t depends only on the Hermitian structure of L, τ depends on the profile as well. By a customary abuse of notation, t and τ are regarded as variables in the intervals (t_1, t_2) and I, respectively.

Define $\pi : L' \to I \times M$ by $\pi = (\tau, p)$. Each fibre of π is an orbit of the S^1-action, so circle-invariant tensors on L' may be identified with tensors on $I \times M$; this will henceforth be done freely, with pullback π^* suppressed. These identifications are diagrammed in Figure 2.1 (a); the vertical maps on the right are projections. The functions μ, s, and f of equation (2.2) are defined on (t_1, t_2), while φ and other functions of geometric interest are defined on I.

Two instances of identification by π deserve immediate mention. First, the Euler vector field $-JX$ on L' pushes forward to $\varphi(\tau)\frac{\partial}{\partial\tau}$ on $I \times M$, cf. equation (2.18) below. The second example arises from the 1-parameter family $\omega_M - \tau\,\gamma$ of Kähler forms on M, regarded as a $(1,1)$-form $\omega_M(\tau)$ on $I \times M$, see Figure 2.1 (b). The family of Ricci forms $\rho_M(\tau)$, and scalar curvature functions $\sigma_M(\tau)$, are similarly regarded as living on $I \times M$. *These tensors depend only on the horizontal data.* If a profile is specified in addition, then there is a map $\pi : L' \to I \times M$ as above, and each of these tensors is identified with an S^1-invariant tensor on L'.

$$
\begin{tikzcd}
 & & M \\
L' \arrow[urr, "p"] \arrow[rr, "\pi"] \arrow[drr, "\tau"] \arrow[d, "t"] & & I \times M \arrow[u] \arrow[d] \\
(t_1, t_2) \arrow[rr, "\mu"] & & I
\end{tikzcd}
$$

Figure 2.1 (a): Relation between maps in the momentum construction

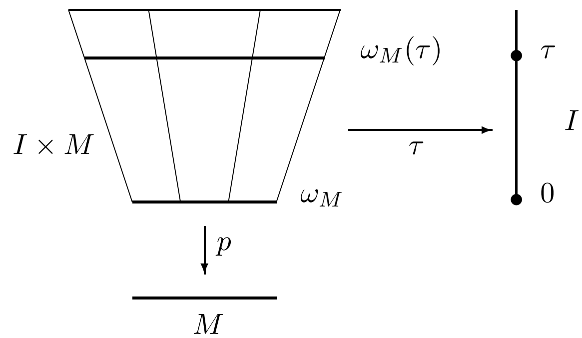

Figure 2.1 (b): Symplectic reduction of g_φ

The Laplacian, Ricci and scalar curvature

Definition 2.1 The *curvature endomorphism* B is $\omega_M^{-1}\gamma$, the symmetric endomorphism of $T^{1,0}M$ obtained by raising the second index of γ. Similarly the *Ricci endomorphism* ϱ of $T^{1,0}M$ is defined by raising the second index of the Ricci form ρ_M. Put

$$
\begin{aligned}
&(2.6) \qquad & Q(\tau) &= \frac{\omega_M(\tau)^m}{\omega_M^m} &&= \det(\mathrm{I} - \tau \mathrm{B}) \\
&(2.7) \qquad & R(\tau) &= \mathrm{tr}_{\omega_M(\tau)} \rho_M &&= \mathrm{tr}\Big[(\mathrm{I} - \tau\mathrm{B})^{-1} \varrho\Big] \\
& & P(\tau) &= 2QR(\tau)
\end{aligned}
$$

Note that P and Q are smooth functions on $I \times M$, with polynomial dependence upon τ, while R may be thought of as a rational function on I, with coefficients depending smoothly on z in M. The notation reflects the fact that when the horizontal data are σ-constant, these functions depend only on τ, i.e. are constant on M for every $\tau \in I$.

On a Kähler manifold, if $g_{i\bar{j}}$ are the components of the metric in local holomorphic coordinates, then the Ricci form is given locally by $\rho = -\sqrt{-1}\partial\bar{\partial} \log V$, where $V = \det(g_{i\bar{j}})$. For a bundle-adapted metric, there is a choice of coordinates such that the quantities V_φ and V_M satisfy

$$
V_\varphi = (\varphi Q)(\tau)\, V_M. \tag{2.8}
$$

The Ricci form of ω_φ is therefore given by

$$
\rho_\varphi = p^* \rho_M - \sqrt{-1}\partial\bar{\partial} \log \varphi Q(\tau), \tag{2.9}
$$

and the scalar curvature is found by taking the trace:

$$
\sigma_\varphi = R(\tau) - \Box_\varphi \log \varphi Q(\tau), \tag{2.10}
$$

$\Box_\varphi$ being the $\bar\partial$-Laplacian of ω_φ. This Laplacian has a reasonably pleasant expression in terms of $\Box_{\omega_M(\tau)}$; for each smooth function ψ on $I \times M$,

$$\Box_\varphi \psi = \Box_{\omega_M(\tau)}\psi(\tau,\cdot) + \frac{1}{2Q}\frac{\partial}{\partial\tau}\left[\varphi Q(\tau)\,\frac{\partial\psi}{\partial\tau}\right]. \tag{2.11}$$

Applying this to the function $\psi = \log(\varphi Q)$ and combining with (2.10) yields

$$\sigma_\varphi = R(\tau) - \Box_{\omega_M(\tau)}\log Q(\tau) - \frac{1}{2Q}\frac{\partial^2}{\partial\tau^2}(\varphi Q)(\tau), \tag{2.12}$$

since $\Box_{\omega_M(\tau)}\log\varphi Q(\tau) = \Box_{\omega_M(\tau)}\log Q(\tau)$, φ being independent of $z \in M$. The first two terms together make up the scalar curvature $\sigma_M(\tau)$ of $\omega_M(\tau)$, so

$$\sigma_\varphi = \sigma_M(\tau) - \frac{1}{2Q}\frac{\partial^2}{\partial\tau^2}(\varphi Q)(\tau). \tag{2.13}$$

If the data are σ-constant, (2.9) and (2.12) simplify as follows:

$$\rho_\varphi = p^*\rho_M + \frac{1}{2Q}(\varphi Q)'(\tau)\,p^*\gamma - \frac{1}{2\varphi}\left[\frac{1}{Q}(\varphi Q)'\right]'(\tau)\,d\tau\wedge d^c\tau \tag{2.14}$$

$$\sigma_\varphi = R(\tau) - \frac{1}{2Q}\frac{\partial^2}{\partial\tau^2}(\varphi Q)(\tau). \tag{2.15}$$

These, together with the expression

$$\omega_\varphi = p^*\omega_M - \tau\,p^*\gamma + \frac{1}{\varphi}\,d\tau\wedge d^c\tau, \tag{2.16}$$

are the main formulae that will be needed in Section 3 in the construction of metrics with prescribed scalar curvature by ODE methods.

2.2 Proofs

We shall now work through the proofs of the statements in the summary just given. We begin by fixing some notation and recalling some basic formulae.

Adapted coordinates

Let $V \subset M$ be a coordinate chart over which L is trivial. Then there exists a *line bundle chart*, namely, a local coordinate system $z^0, z^1, \ldots, z^m$ for L in which $z^0 = \rho e^{i\theta}$ is a fibre coordinate and $z = (z^1, \ldots, z^m)$ are (pullbacks of) coordinates on M. Greek indices run from 1 to m, while Latin indices run from 0 to m.

In such a chart, there is a smooth, positive function $h : V \to \mathbf{R}$ such that $r = z^0\bar z^0\,h(z)$; under change of chart, the local function h is multiplied by the norm squared of a non-vanishing local holomorphic function in V. In a line bundle chart, the Euler vector field is given by

$$\Upsilon = z^0\frac{\partial}{\partial z^0} = \frac{1}{2}\left(\rho\frac{\partial}{\partial\rho} - \sqrt{-1}\frac{\partial}{\partial\theta}\right),$$

while twice the imaginary and real parts are described variously as

$$X = -2\,\mathrm{Im}\,\Upsilon = \sqrt{-1}(\Upsilon - \overline{\Upsilon}) = \frac{\partial}{\partial\theta}, \qquad H = -JX = 2\,\mathrm{Re}\,\Upsilon = (\Upsilon + \overline{\Upsilon}) = \rho\frac{\partial}{\partial\rho}.$$

In particular this gives a local formula for the normalized generator X of the S^1 action on L. It is sometimes convenient to use the fibre coordinate $w^0 = \log z^0 =: \zeta + i\theta$, in which case $\Upsilon = \partial/\partial w^0$.

The level sets of r are real hypersurfaces in L, and their tangent spaces are the horizontal spaces of the Hermitian connection $\Theta = \partial \log r = 2\partial t$ of (L, h). For each point x of M, there exists a line bundle chart (z^0, z) such that $z^\alpha(x) = 0$ for $1 \le \alpha \le m$ and $\partial_\alpha r = 0$ on the fibre L_x. Such a coordinate system is said to be *adapted* to (L, h) at $x \in M$. In adapted coordinates at x, the connection form Θ is equal to $dw^0 = dz^0/z^0$ along the fibre L_x.

Sign conventions and Levi forms

The wedge product is normalized so that interior multiplication is a (graded) derivation: If V is a vector and ξ and η are 1-forms, then $i_V(\xi \wedge \eta) = \langle V, \xi\rangle\eta - \langle V, \eta\rangle\xi$. Extensive use will be made of the real operators $d = \partial + \bar\partial$ and $d^c = \sqrt{-1}(\bar\partial - \partial)$, and of the useful formulae

$$dd^c\, u = 2\sqrt{-1}\partial\bar\partial u, \qquad du \wedge d^c u = 2\sqrt{-1}\partial u \wedge \bar\partial u,$$

which hold for every smooth function u. Finally if g has components $(g_{i\bar j})$ in local holomorphic coordinates (z^i), then $\omega = \frac{\sqrt{-1}}{2} g_{i\bar j} dz^i \wedge d\bar z^j$.

The curvature form γ of an Hermitian line bundle satisfies $-dd^c t = -\sqrt{-1}\partial\bar\partial \log r = p^*\gamma$. Combining this with the chain rule gives the simple but important formula

$$dd^c u(t) = u''(t)\, dt \wedge d^c t - u'(t)\, p^*\gamma \tag{2.17}$$

when u is a smooth function of one variable. This is often used in the form

$$dd^c\psi(\tau) = -(\varphi\psi')(\tau)\, p^*\gamma + \frac{1}{\varphi}(\varphi\psi')'(\tau)\, d\tau \wedge d^c\tau, \tag{2.18}$$

which follows from (2.17) and the first of (2.3).

Proof of Proposition 1.4

Referring to the momentum construction, especially formulae (2.1)–(2.4), it must be shown that g_φ is a Kähler metric (i.e. that $\omega_\varphi > 0$), and that the moment map is given by τ. Expanding $dd^c f$ using (2.18) gives (2.16):

$$\omega_\varphi = p^*\omega_M - \tau\, p^*\gamma + \varphi\, dt \wedge d^c t = p^*\omega_M - \tau\, p^*\gamma + \frac{1}{\varphi}\, d\tau \wedge d^c\tau.$$

Since $\omega_M - \tau\gamma$ and φ are both positive on I by hypothesis, it follows that ω_φ is positive. Now

$$i_X(dt) = 0 = i_X\Big(\omega_M(\tau)\Big),$$

so $i_X\omega_\varphi = -\varphi\, dt \wedge i_X(d^c t) = -\varphi\, dt$ from the formula

$$d^c t = \frac{\sqrt{-1}}{2}\left(\frac{d\bar{z}^0}{\bar{z}^0} - \frac{dz^0}{z^0}\right) + \frac{1}{2} d^c \log h$$

in a line-bundle chart. But $\varphi\, dt = d\tau$ by (2.3), so τ is a choice of moment map. A change of τ_0 just adds a constant to t (i.e. changes L' by a homothety) and does not change the isometry class of ω_φ.

In adapted coordinates,

$$\omega_\varphi = \frac{\sqrt{-1}}{2}\left(\frac{\varphi(\tau)}{2z^0\bar{z}^0}\, dz^0 \wedge d\bar{z}^0 + \Big[g_M(\tau)\Big]_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta\right) \tag{2.19}$$

along a fibre. In terms of globally defined functions, the metric splits *along a fibre* into

$$\omega_{\text{fibre}} + \omega_{\text{horiz}} = ds \wedge d^c s + \omega_M(\tau) \tag{2.20}$$

The expression (2.5) for the fibre metric follows at once, as does the fact that s is the geodesic distance in the fibre.

The Ricci form

From (2.19), the volume form is

$$\frac{\omega_\varphi^{m+1}}{(m+1)!} = (\varphi Q)(\tau)\ \det\Big[(g_M)_{\alpha\bar{\beta}}\Big] \frac{1}{z^0\bar{z}^0}\left(\Big(\frac{\sqrt{-1}}{2}\Big)^{m+1} \prod_{i=0}^{m} dz^i \wedge d\bar{z}^i\right), \tag{2.21}$$

from which (2.8) follows immediately.

Denote by ∂_M and $\bar{\partial}_M$ the ∂ and $\bar{\partial}$-operators on M; it is necessary here to distinguish them from the corresponding operators on L, which we continue to denote by ∂ and $\bar{\partial}$. For every smooth function u on $I \times M$,

$$dd^c u = d_M d^c_M u - \varphi \frac{\partial u}{\partial \tau}\, \gamma + \frac{1}{\varphi}\frac{\partial}{\partial \tau}\left[\varphi \frac{\partial u}{\partial \tau}\right] d\tau \wedge d^c\tau + \text{cross-terms.}$$

(The cross-terms take the form $\bar{\partial}\tau \wedge \partial_M(\partial u/\partial\tau) +$ complex conjugate.) In order to take the trace with respect to ω_φ, either work locally or multiply by ω_φ^m. For the latter,

$$\omega_\varphi^m = \omega(\tau)^m + m\,\omega(\tau)^{m-1} \cdot \frac{1}{\varphi}\, d\tau \wedge d^c\tau.$$

Wedging with $dd^c u$, the cross-terms drop out, so upon division by ω_φ^{m+1},

$$\Box_\varphi u = \Box_{\omega_M(\tau)} u(\tau, \cdot) + \frac{1}{2}\left(\frac{\partial}{\partial \tau}\Big(\varphi \frac{\partial u}{\partial \tau}\Big) - \Big(\mathrm{tr}_{\omega_M(\tau)}\gamma\Big)\, \varphi \frac{\partial u}{\partial \tau}\right).$$

The right-hand side of this is simplified by the following observation: If the eigenvalues of B are denoted $\beta_\nu(z)$, $\nu = 1, \ldots, m$, then $Q(\tau) = \prod_\nu (1 - \tau\,\beta_\nu(z))$, so

$$\mathrm{tr}_{\omega_M(\tau)}\gamma = \mathrm{tr}[(\mathrm{I} - \tau\mathrm{B})^{-1}\mathrm{B}] = \sum_{\nu=1}^{m} \frac{\beta_\nu(z)}{1 - \tau\,\beta_\nu(z)} = -\frac{\partial}{\partial\tau}(\log Q). \tag{2.22}$$

Equations (2.11) and (2.12) follow immediately, while (2.13) follows from the observation

$$\Box_{\omega_M(\tau)} \log \varphi Q = \Box_{\omega_M(\tau)}(\log\varphi + \log Q) = \Box_{\omega_M(\tau)} \log Q$$

(φ being pulled back by the first projection $I \times M \to I$) and the formulae

$$\begin{aligned} -\sqrt{-1}\partial_M\bar{\partial}_M \log Q &= \rho_{\omega_M(\tau)} - \rho_M & (2.23)\\ -\Box_{\omega_M(\tau)} \log Q &= \mathrm{tr}_{\omega_M(\tau)}(\rho_{\omega_M(\tau)} - \rho_M) = \sigma_M(\tau) - R(\tau). & (2.24)\end{aligned}$$

Total geodesy of fibres

It is well-known that the fibres of L' are totally geodesic with respect to ω. The proof given here serves as an excuse to calculate the Levi-Civita connection of g.

Proposition 2.2 *Let g_φ be a bundle-adapted metric on $L' \subset L$. Then each fibre of L' is totally geodesic with respect to g_φ.*

Proof It suffices to calculate the Levi-Civita connection D of g_φ and show that $D_{\partial_0}\partial_0$ is tangent to the fibre. Let (z^0, z) be a line bundle chart, $w^0 = \log z^0$, and let ∂_α denote partial differentiation. Recall that the connection form of (L, h) is equal to $\Theta = \partial \log r$. With respect to the coordinates (w^0, z), the vector-valued $(1,0)$-form Θ is given by the column

$$[\Theta_i]^{\mathrm{t}} = [\,1 \quad \Theta_\alpha\,]^{\mathrm{t}}, \qquad \Theta_\alpha = h^{-1}\partial_\alpha h,$$

and $\bar{\partial}(\Theta_\alpha\, dz^\alpha) = p^*\gamma$. The components of g_φ are given by the Hermitian $(1+m)\times(1+m)$ block matrix

$$G = \begin{bmatrix} g_{0\bar{0}} & g_{0\bar{\beta}} \\ g_{\alpha 0} & g_{\alpha\bar{\beta}} \end{bmatrix} = 2\varphi(\tau) \begin{bmatrix} 1 & [\Theta_{\bar{\beta}}]^{\mathrm{t}} \\ [\Theta_\alpha] & [\Theta_\alpha][\Theta_{\bar{\beta}}]^{\mathrm{t}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & [g_M(\tau)]_{\alpha\bar{\beta}} \end{bmatrix}. \tag{2.25}$$

The inverse matrix G^{-1} is found by (block) row-reduction:

$$G^{-1} = \frac{1}{2\varphi(\tau)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} [\Theta_{\bar{\beta}}]^{\mathrm{t}}\,[g_M(\tau)]^{\bar{\beta}\alpha}\,[\Theta_\alpha] & -[\Theta_{\bar{\beta}}]^{\mathrm{t}}\,[g_M(\tau)]^{\bar{\beta}\alpha} \\ -[g_M(\tau)]^{\bar{\beta}\alpha}\,[\Theta_\alpha] & [g_M(\tau)]^{\bar{\beta}\alpha} \end{bmatrix}. \tag{2.26}$$

The Levi-Civita connection form of g is represented in a line bundle chart by the matrix-valued $(1,0)$-form $G^{-1}\partial G$. A short calculation shows that in adapted coordinates,

$$G^{-1}\partial G = \begin{bmatrix} (1/2)\varphi'(\tau)\partial t & \partial[\Theta_{\bar{\beta}}]^{\mathrm{t}} \\ \varphi(\tau)[g_M(\tau)]^{\bar{\beta}\alpha}\partial[\Theta_\alpha] & [g_M(\tau)]^{\bar{\beta}\nu}[g_M(\tau)]_{\alpha\bar{\beta}} \end{bmatrix}$$

along the fibre. Evaluating this matrix-valued $(1,0)$-form on the tangent vector $\partial_0 = \partial/\partial w^0$ at a point of the fibre gives the representation (with respect to the frame $\{\partial_i\}_{i=0}^n$) of the covariant derivative D_{∂_0} in the fibre direction. The resulting matrix has first column $[\,(1/2)\varphi'(\tau) \quad 0 \quad \cdots \quad 0\,]^{\mathrm{t}}$, which implies the covariant derivative $D_{\partial_0}\partial_0$ is tangent to the fibre, i.e. that the fibre is totally geodesic. $\square$

2.3 Completeness and Extendability of Fibre Metrics

In this section, the completeness properties of the metric g_φ are given in terms of the boundary behaviour of φ. Because each fibre is totally geodesic by Proposition 2.2, g_φ is complete iff

- The metric $g_M(\tau)$ is complete for every $\tau \in I$, and
- The fibre metric g_{fibre} is complete.

The first condition will be assumed, so the task is to relate the second to the boundary behaviour of the profile. By equation (2.2), completeness is guaranteed by divergence of the s integral, though divergence is not necessary since a fibre metric may extend smoothly to the origin. In any case, criteria in terms of order of vanishing are easier to work with. By definition, a profile is smooth on the closure of I, so at each finite endpoint α there is an expansion

$$\varphi(\tau) = a_\ell(\tau - \alpha)^\ell + O((\tau - \alpha)^{\ell+1}), \qquad a_\ell \neq 0, \tag{2.27}$$

with $\ell \geq 0$ an integer. For convenience, it will also be assumed that φ is asymptotic to an integer power of τ as $\tau \to \infty$. This is no real loss, since the profiles of greatest interest are *rational* functions, which arise when g_φ has constant scalar curvature (or is formally extremal). For such profiles, the relation between boundary behaviour and completeness is very simple:

Proposition 2.3 *Let $\varphi : I \to \mathbf{R}$ be a profile as above. Then the associated fibre metric is complete if and only if one of the following conditions holds at each endpoint of I:*

- *Finite Endpoint(s)*
  - (i) *The profile φ vanishes to first order, and $|\varphi'| = 2$ at the endpoint; or*
  - (ii) *The profile vanishes to order at least two.*
- *Infinite Endpoint(s)*
  - (iii) *The profile grows at most quadratically, i.e. there is a positive constant K such that $\varphi(\tau) \leq K\tau^2$ for $|\tau| \gg 0$.*

Apart from the proof of this result, this section is devoted to the classification of S^1-invariant metrics of constant Gaussian (equivalently scalar) curvature on S^1-invariant domains of $\mathbf{P}^1$. The two subsections may be read in either order; it is hoped that Table 2.2 below will illuminate the proof of Proposition 2.3.

Proof of Proposition 2.3

Lemma 4.1 below asserts there is no loss of generality in taking the lower endpoint of I to be 0; this is assumed here for convenience. The boundary condition $\varphi(0) = 0$ is necessary for completeness of g_{fibre}; if $\varphi(0) > 0$ then the s integral in (2.2) converges at $\tau = 0$, so the distance to the level set $\{\tau = 0\}$ is finite, while the length of the corresponding S^1-orbit is $2\pi\sqrt{\varphi(0)} > 0$. This is impossible for a complete metric.

Hence φ vanishes to order $\ell \geq 1$, the t integral diverges at $\tau = 0$, and the level set $\{\tau = 0\}$ intersects the fibre at the origin. If $\ell \geq 2$ the s integral is also divergent, so the origin is at infinite distance, i.e. the fibre metric is complete. If $\ell = 1$, the s-integral is convergent and more work is needed to determine whether the fibre metric extends smoothly to the level set $\{r = 0\}$.

In a line bundle chart, $r = h\,|z^0|^2$, and by (2.5) the fibre metric is

$$g_{\text{fibre}} = \varphi(\tau)\left|\frac{dz^0}{z^0}\right|^2 = \left[\frac{\varphi(\tau)}{r}\right]\, h\,|dz^0|^2.$$

Since the horizontal part is smooth, g_φ is smooth at $r = 0$ iff $\varphi(\tau)/r$ has a finite, positive limit as $\tau \to 0$. (This limit makes sense, for (2.2) gives t and hence r as a function of τ.) Using (2.2) to find t, with $\varphi(\tau) = a_1\tau + O(\tau^2)$, gives $\log\tau = \text{constant} + a_1 t + \cdots$, or $\tau = a\,r^{a_1/2} + \cdots$ for some $a > 0$. Thus $\varphi(\tau)/r$ has a finite, positive limit as $r \to 0$ iff $a_1 = 2$, i.e. $\varphi(0) = 0$, $\varphi'(0) = 2$. This completes the analysis at the lower endpoint of I. The analysis at the upper endpoint b is similar when $b < \infty$; in particular, the fibre metric extends to infinity in the fibre iff $\varphi(b) = 0$ and $\varphi'(b) = -2$.

It remains to consider the case $(0,\infty) \subset I$. Completeness of the fibre metric, i.e. unboundedness of the distance to the level set $\{t = t_2\}$, limits the growth of φ at ∞ via the relation

$$\lim_{\tau\to\infty}\int^{\tau}\frac{dx}{\sqrt{\varphi(x)}} = \infty. \tag{2.28}$$

The quadratic growth condition (iii) is immediate; this completes the proof of Proposition 2.3. □

The boundary behaviour of φ and completeness properties of g_{fibre} are summarized in Table 2.1.

Geometry of fibre metrics

In this section, the classification of complete S^1-invariant metrics on subsets of $\mathbf{P}^1$ is considered from the point of view of the momentum construction. The results are summarized in Table 2.2 below, which shows clearly how much simpler these metrics look in 'momentum coordinates' than in the standard complex coordinate z.

Since the scalar curvature is given in terms of the profile as $-\varphi''/2$, constancy of the scalar curvature implies that the profile is quadratic. The list of profiles, together with the intervals I^0 on which they are positive, appear in the first two columns of the table. As usual, the normalization $\inf I = 0$ has been used in all but cases (v) and (vi), where $I = \mathbf{R}$. Note that I^0 is the *interior* of the image I of the moment map.

Profile Near $\tau = 0$	Distance to $\tau = 0$	End Geometry	Complete/Smooth
$\varphi(\tau) = 2\tau + O(\tau^2)$	Finite	Smooth Extension	Yes
$0 < \varphi'(0) \neq 2$	Finite	Point Singularity	No
$\varphi(\tau) \leq K\tau^2$	Infinite	Finite-area Cusp	Yes

Profile Near $\tau = \infty$	Distance to $\tau = \infty$	End Geometry	Complete
$\varphi(\tau) \to 0$	Infinite	Infinite-area Cusp	Yes
$\varphi(\tau) \to \text{const} > 0$	Infinite	Cylindrical	Yes
$\varphi(\tau) \sim K\tau$	Infinite	Planar/Conical	Yes
$\varphi(\tau) \sim K\tau^2$	Infinite	Hyperbolic	Yes
$\varphi(\tau) \geq K\tau^3$	Finite		No

Table 2.1: Completeness and boundary/asymptotic behavior for rational profiles.

Performing the t integral in each case yields t—and hence $r = |z|^2$—as a function of τ. From these formulae, the r interval in the third column follows at once. The next three columns follow either directly or from Proposition 2.3. Finally, the metric in question is named, its scalar curvature is given, and the conformal factor $[\varphi(\tau)/r]$ is written explicitly as a function of r. The latter is obtained by inverting the dependence of r upon τ and substituting in the formula for φ.

Remark 2.4 (iv) is the finite-area hyperbolic cusp. In (vi) the annulus is determined, up to conformal equivalence, by the ratio of the inner and outer radii and hence by $c\alpha$. In the examples (i)–(iii), with $\varphi'(0) = 2$, metrics with conical singularity at the origin arise by changing the coefficient of τ to $a \neq 2$. □

It is instructive to compare these metrics with surfaces of revolution in $\mathbf{R}^3$. Let ξ be a positive function, whose graph sweeps out a surface of revolution. The area and length elements are given in terms of the independent variable y as

$$d\tau = \xi(y)\sqrt{1 + \xi'(y)^2}\, dy, \qquad ds = \sqrt{1 + \xi'(y)^2}\, dy.$$

The profile is the length squared of the vector field X, or $\varphi(\tau) = \xi(y)^2$. Differentiating and using the previous equations gives

$$\varphi'(\tau) = \frac{2\xi'(y)}{\sqrt{1 + \xi'(y)^2}}, \qquad \text{or} \qquad \xi'(y) = \frac{\varphi'(\tau)}{\sqrt{4 - \varphi'(\tau)^2}},$$

which implies $|\varphi'(\tau)| \leq 2$, with equality iff $|\xi'(y)| = \infty$: The fibre metric embeds as a surface of revolution iff the profile is not too steep. Comparing with the profiles in the table, one notes the well-known facts that the portion of the cusp (iv) corresponding to the momentum sub-interval $(0, 1/c^2)$ embeds as a surface of revolution in $\mathbf{R}^3$ (the pseudosphere), while no annulus in the Poincaré disc (iii) arises in this way.

It is also worth noting that intuition deriving from surfaces of revolution can be misleading. Taking $I = [0, \infty)$ and $\varphi(\tau) = 2\tau + \tau^3$ gives an incomplete metric of infinite area on the disk!

	I^0	$\varphi(\tau)$	r-range	Distance to $\tau = a$ $\tau = b$	Area near $\tau = a$ $\tau = b$	Metric σ $\frac{\varphi(\tau)}{r}$	Domain
(i)	$(0, 2c^{-2})$	$2\tau - c^2\tau^2$	$[0, \infty]$	finite finite	finite finite	Fubini-Study c^2 $\frac{4}{c^2(1+r)^2}$	$\mathbf{P}^1$
(ii)	$(0, \infty)$	2τ	$[0, \infty)$	finite infinite	finite infinite	flat plane 0 1	$\mathbf{C}$
(iii)	$(0, \infty)$	$2\tau + c^2\tau^2$	$[0, 1)$	finite infinite	finite infinite	Poincaré disc $-c^2$ $\frac{4}{c^2(1-r)^2}$	Δ
(iv)	$(0, \infty)$	$c^2\tau^2$	$(0, 1)$	infinite infinite	finite infinite	hyperbolic cusp $-c^2$ $\frac{4}{c^2\,r(\log r)^2}$	$\Delta^\times$
(v)	$\mathbf{R}$	α^2	$(0, \infty)$	infinite infinite	infinite infinite	flat cylinder 0 $\frac{\alpha^2}{r}$	$\mathbf{C}^\times$
(vi)	$\mathbf{R}$	$\alpha^2 + c^2\tau^2$	$(e^{-\frac{\pi}{c\alpha}}, e^{\frac{\pi}{c\alpha}})$	infinite infinite	infinite infinite	hyperbolic annulus $-c^2$ $\frac{\alpha^2}{r\cos^2(c\alpha\log r/2)}$	Annulus

Table 2.2: Classification of complete, circle-invariant metrics with constant scalar curvature on domains in $\mathbf{P}^1$ by momentum data.

3 Metrics of Infinite Volume

This section is devoted to the proofs of Theorems B, C, and D. Section 3.1 deals with Theorems B and C, while Section 3.2 with Theorem D. The proofs are separated in this way because of the behaviour of the family of Kähler forms $\omega_M(\tau)$; in Theorems B and C, $\omega_M(0)$ is non-degenerate, whereas in Theorem D the family $\{\omega_M(\tau)\}$ drops rank at $\tau = 0$, leading to a metric on a partial blow-down of L.

3.1 Metrics on Line Bundles

Let $\{p : (L, h) \to (M, \omega_M),\ I\}$ be σ-constant horizontal data with (L, h) not flat, and with $I = [0, \infty)$ or $(0, \infty)$. In particular, $\gamma \leq 0$, so $\omega_M(\tau) = \omega_M - \tau\gamma$ is a Kähler form on M for all $\tau \geq 0$. Further, the metric $g_M(\tau)$ is complete by hypothesis because g_M is assumed to be complete and $-\tau\gamma$ is positive semidefinite if $\tau \in I$. As in Section 2, define functions Q, $R : I \to \mathbf{R}$ by

$$Q(\tau) = \det(1 - \tau\mathrm{B}), \qquad R(\tau) = \mathrm{tr}\Big[(1 - \tau\mathrm{B})^{-1}\varrho\Big].$$

Because the data are σ-constant, $Q(\tau)$ is the (constant) scale factor for the volume form of $\omega_M(\tau)$ relative to ω_M and is a polynomial in τ with only negative roots, while $R(\tau) = \sigma_M(\tau)$ is the scalar curvature of $g_M(\tau)$ and is a rational function that is bounded below on I. Indeed, R has an asymptotic value $R(\infty)$ as $\tau \to \infty$, which may be interpreted as the trace of ϱ restricted to the 0-eigenbundle of B. By Corollary A.1, the scalar curvature of ω_φ is

$$\sigma_\varphi = R(\tau) - (1/2Q)(\varphi Q)''(\tau). \tag{3.1}$$

Now let σ be a function on I. The problem of prescribing the scalar curvature of ω_φ is given by the equation $\sigma = \sigma_\varphi$, which has the solution

$$(\varphi Q)(\tau) = (\varphi Q)(0) + (\varphi Q)'(0)\tau + 2\int_0^\tau (\tau - x)\Big(R(x) - S(x)\Big)Q(x)\,dx \tag{3.2}$$

in terms of the initial data $\varphi(0)$ and $\varphi'(0)$. The momentum construction yields a metric of infinite fibre area iff φ is positive on $(0,\infty)$. This metric is complete if φ grows at most quadratically at ∞ and satisfies the boundary conditions given in Proposition 2.3: $\varphi(0) = 0$, and either $\varphi'(0) = 2$ or $\varphi'(0) = 0$. The first case is the one needed for Theorem B, the second for Theorem C. These will now be considered in turn.

Proof of Theorem B

Setting $\sigma = c$ (constant) and using the initial conditions $\varphi(0) = 0$, $\varphi'(0) = 2$ in (3.2) gives

$$\varphi(\tau) = \frac{2}{Q(\tau)}\left(\tau + \int_0^\tau (\tau - x)\Big(R(x) - c\Big)Q(x)\,dx\right). \tag{3.3}$$

The notation $\varphi_c(\tau)$ or $\varphi(\tau, c)$ will be used when the dependence of the profile on c is being emphasized. Define the set $\mathrm{J} \subset \mathbf{R}$ of "allowable scalar curvatures" by

$$\mathrm{J} = \{c \in \mathbf{R} \mid \varphi_c(\tau) > 0 \text{ for all } \tau > 0\}.$$

In words, J is the set of c for which equation (3.3) defines a momentum profile on I.

Lemma 3.1 *There is a $c_0 \in \mathbf{R}$ such that either $\mathrm{J} = (-\infty, c_0)$ or $\mathrm{J} = (-\infty, c_0]$.*

Proof The function R is bounded on I, so the integrand in (3.3) is positive for $c \ll 0$, implying $\varphi_c > 0$ on $(0,\infty)$ for $c \ll 0$. In particular, J is non-empty. If $\tau > 0$, then

$$\frac{\partial \varphi}{\partial c}(\tau, c) = -\frac{2}{Q(\tau)}\int_0^\tau (\tau - x)Q(x)\,dx < 0,$$

i.e. $\varphi(\tau, c)$ is strictly decreasing with respect to c. Consequently, if $c \in \mathrm{J}$ and $c' < c$, then $c' \in \mathrm{J}$. Finally, J is bounded above since by (3.3), φ_c is not everywhere positive for $c > R(\infty)$. In summary, J is a half-line, unbounded below. Set $c_0 = \sup \mathrm{J} \le R(\infty)$. $\square$

Since the initial condition ensures that φ_c is positive for sufficiently small positive τ, the momentum construction yields a metric which for notational convenience will be denoted g_c.

Lemma 3.2 *If $c < c_0$ then g_c is a complete metric on $\Delta(L)$, and g_{c_0} is a complete metric on L.*

Proof By Proposition 2.3 and Table 2.1, completeness of the metric is equivalent to 'at most quadratic' growth of φ_c at ∞, provided the profile is positive on $(0, \infty)$. To establish the growth condition, it is easiest to write $R - c$ as a constant plus a rational function R_0 vanishing at ∞:

$$R(\tau) - R(\infty) =: R_0(\tau), \quad c - R(\infty) =: \tilde{c}, \qquad \text{so} \quad R(\tau) - c = R_0(\tau) - \tilde{c}.$$

Define polynomials

$$P_1(\tau) = \tau + \int_0^\tau (\tau - x) R_0(x) Q(x)\, dx, \qquad P_2(\tau) = \int_0^\tau (\tau - x) Q(x)\, dx;$$

Because R_0 vanishes at ∞, $\deg P_1 \le 1 + \deg Q$, and $\deg P_2 = 2 + \deg Q$. From (3.3),

$$\varphi(\tau, c) = \frac{2}{Q(\tau)}\Big(P_1(\tau) - \tilde{c} P_2(\tau)\Big), \tag{3.4}$$

so if $\tilde{c} = 0$, then $\varphi_c(\tau) \le K\tau$ as $\tau \to \infty$, while $\varphi_c(\tau) \sim K\tau^2$ if $\tilde{c} < 0$, i.e. if $c < R(\infty)$. Since $c_0 \le R(\infty)$, it follows that each profile φ_c with $c < c_0$ gives rise to a complete fibre metric, hence to a complete metric g_c. Because φ_c grows quadratically, the t integral converges as $\tau \to \infty$, so up to homothety g_c lives on the unit disk bundle $\Delta(L)$.

It remains to investigate the borderline case. First observe that every profile φ_c with $c < c_0$ is bounded away from zero except near $\tau = 0$ (since $\varphi_c'(0) = 2$, and φ_c is positive on $(0, \infty)$ and has infinite limit as $\tau \to \infty$). By the proof of Lemma 3.1, $\varphi(\tau, c)$ is decreasing in c. Since c_0 is the supremum of c for which φ_c is positive on $(0, \infty)$, it follows that φ_{c_0} is non-negative on I by continuity of $\varphi(\tau, c)$ in c. The borderline profile is not identically zero, since $\varphi_{c_0}'(0) = 2$.

Two possibilities occur: φ_{c_0} has a positive zero, or is positive on $(0, \infty)$. In the first case let b be the first positive zero of φ_{c_0}. Then $\varphi_{c_0}'(b) = 0$ as well, for φ_{c_0} is real-analytic and non-negative. Since φ_{c_0} vanishes to order at least two, the t and s integrals diverge, so the associated metric lives on the total space of L and is complete, but has finite-area fibres, see also Table 2.1.

Consider now the second possibility, $\varphi_{c_0} > 0$ on $(0, \infty)$. We claim in this case that $c_0 = R(\infty)$ (i.e. $\tilde{c} = 0$). For if not, the borderline profile is positive and grows quadratically, hence is bounded away from zero except near $\tau = 0$; a glance at (3.4) shows $\tilde{c}$ may be increased slightly, preserving positivity of the profile, but this contradicts the definition of c_0. Hence $\varphi_{c_0}(\tau) = (P_1/Q)(\tau) \le K\tau$ and again g_{c_0} lives on L. □

The dichotomy at $c = c_0$ is summarized as follows:

- φ_{c_0} has a positive zero in I and yields a metric with fibrewise finite area.
- φ_{c_0} is positive on $(0, \infty)$ and yields a complete metric of infinite fibre area;

As shown above, the second alternative implies $c_0 = R(\infty)$ (so $c_0 < R(\infty)$ implies the first alternative, see Figure 3.1), but this is the only general conclusion that can be drawn. Further, it is not necessarily true that $c_0 = 0$ (since generally $R(\infty) \ne 0$, for example); this issue is addressed

in detail below. However, if $\gamma < 0$, or if the construction yields a metric that is Einstein, then the borderline metric is scalar-flat, while the others have negative curvature.

The borderline constant c_0 has an alternative interpretation. Consider the equation $\varphi(\tau, c) = 0$. From (3.4), this level set is the graph of the rational function $C : I \to \mathbf{R}$ defined by

$$C(\tau) = R(\infty) + \frac{P_1(\tau)}{P_2(\tau)}.$$

The degree of P_1 is less than the degree of P_2, so $C(\tau) \to R(\infty)$ as $\tau \to \infty$. Furthermore, P_1 vanishes to order one and P_2 vanishes to order two at $\tau = 0$, so $C(\tau) \to +\infty$ as $\tau \to 0^+$. Hence the function C is bounded below on $(0, \infty)$, and it follows immediately from the definition that $c_0 = \inf\{C(\tau) \mid \tau \in (0, \infty)\}$, see Figure 3.1.

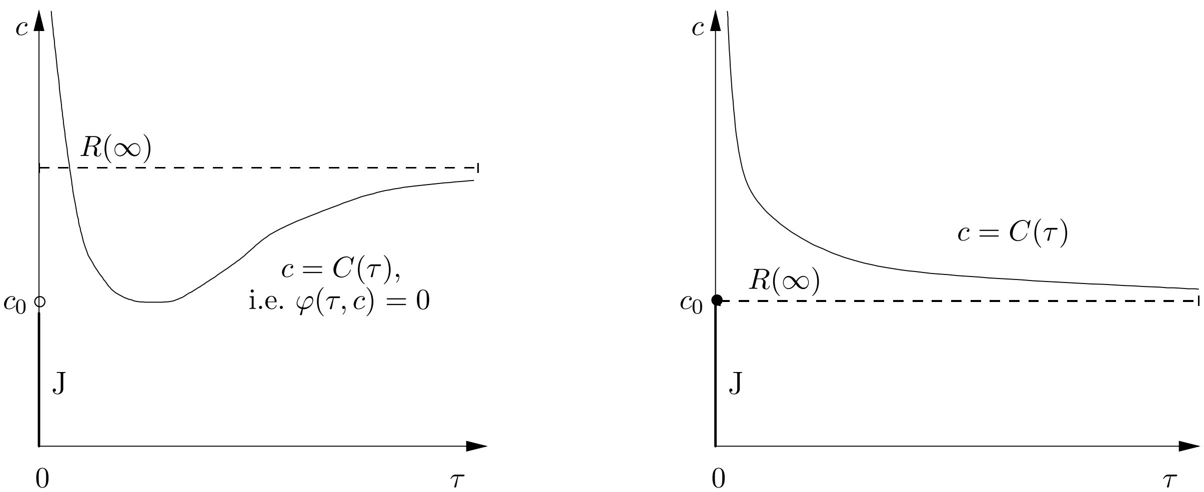

Figure 3.1: The rational function C and the interval J of allowable scalar curvatures

If $c > c_0$, then φ_c is *not* non-negative on $(0, \infty)$, so there is a first positive root b, and the fibre metric has finite area. Three possibilities occur:

1. $\varphi_c'(b) = 0$, and the fibre metric is complete, with a cusp end at $\tau = b$;
2. $\varphi_c'(b) = -2$, and the metric extends smoothly to the $\mathbf{P}^1$-bundle $\widehat{L}$ (but note Remark 3.5 below);
3. $\varphi_c'(b) \neq 0,\ -2$, so the metric is incomplete and has no smooth extension.

Lemma 3.3 *There are at most finitely many $c > c_0$ for which the profile φ_c satisfies one of the first two conditions.*

Proof By (3.4), $\varphi(\tau, c) = 0$ if and only if $\tilde{c} = P_1(\tau)/P_2(\tau)$. Substitution shows that if $\varphi(\tau, c) = 0$, then

$$\frac{\partial\varphi}{\partial\tau}(\tau, c) = 2\frac{P_1(\tau)}{Q(\tau)}\left(\log\frac{P_1}{P_2}\right)'(\tau).$$

This is a non-constant rational function, which takes the values 0 and -2 for at most finitely many values of τ. Consequently, there are at most finitely many pairs (τ, c) satisfying

$$\varphi(\tau, c) = 0 \qquad \text{and} \qquad \frac{\partial\varphi}{\partial\tau}(\tau, c) = 0 \text{ or } -2.$$

In any event, a value $c > c_0$ does not give rise to a metric with infinite volume. $\square$

The final task is to determine when a metric just constructed is Einstein. Equation (2.16) expresses the Kähler form of g_φ in terms of φ and the horizontal data, while (2.14) similarly expresses the Ricci form. Setting $\rho_\varphi = \lambda\,\omega_\varphi$ gives

$$-\Big(\frac{1}{2Q}(\varphi Q)'\Big)' = \lambda \qquad \text{and} \qquad \rho_M + \frac{1}{2Q}(\varphi Q)'(\tau)\,\gamma = \lambda(\omega_M - \tau\gamma).$$

Integrating the first and using the initial conditions $\varphi(0) = 0$, $\varphi'(0) = 2$,

$$\frac{1}{2Q}(\varphi Q)'(\tau) = 1 - \lambda\tau. \tag{3.5}$$

Substituting this back into the second equation,

$$\rho_M + \gamma = \lambda\omega_M.$$

Assume from now on that $\lambda \le 0$, see Remark 3.5 below. Integrating (3.5), again using $\varphi(0) = 0$, gives

$$\varphi(\tau) = \frac{2}{Q(\tau)}\int_0^\tau (1 - \lambda x)Q(x)\,dx, \tag{3.6}$$

which is clearly positive for all $\tau > 0$. Completeness follows since φ grows linearly (if $\lambda = 0$) or quadratically (if $\lambda < 0$), and the scalar curvature is $\lambda(m+1)$. Conversely, a bundle-adapted metric arising in this way is Einstein:

Lemma 3.4 *If the horizontal data are σ-constant and satisfy $\rho_M + \gamma = \lambda\omega_M$ for some $\lambda \le 0$, then the bundle-adapted metric with $c = \lambda(m+1)$ is Einstein-Kähler.*

Proof Write $\rho_M = \lambda\omega_M - \gamma = \lambda(\omega_M - \tau\gamma) - (1 - \lambda\tau)\gamma$. Taking the trace with respect to $\omega_M(\tau)$, using the definition of R and recalling that $\mathrm{tr}_{\omega_M(\tau)}\gamma = -(\partial/\partial\tau)(\log Q)$ by equation (2.22), immediately implies

$$R(\tau) = \lambda m + (1 - \lambda\tau)\frac{Q'(\tau)}{Q(\tau)}.$$

If $c = \lambda(m+1)$, then

$$\Big(R(\tau) - c\Big)Q(\tau) = Q'(\tau) - \lambda\Big(Q(\tau) + \tau Q'(\tau)\Big) = \frac{d}{d\tau}\Big((1-\lambda\tau)Q(\tau)\Big).$$

Integrating twice, using $\varphi(0) = 0$ and $\varphi'(0) = 2$, proves the profile (3.6) coincides with the profile (3.3). $\square$

This completes the proof of Theorem B.

Remark 3.5 When the Einstein constant λ is positive, every completion of $L^\times$ is compact by Myer's Theorem, so there is a non-vacuous boundary condition imposed on (3.5), namely that $\varphi'(b) = -2$ when $\varphi(b) = 0$. Koiso and Sakane have shown this is equivalent to vanishing of the Futaki invariant of the compactification. This boundary condition is never satisfied if the curvature form γ is negative semi-definite unless $\gamma = 0$. To see this, observe that $\varphi(b) = 0$ if and only if

$$\int_0^b (1 - \lambda x)Q(x)\,dx = 0,$$

in which case a short calculation shows that $\varphi'(b) = -2$ if and only if $b\lambda = 2$. Substituting back into (3.6),

$$\varphi(\tau) = \frac{2}{Q(\tau)}\int_0^\tau (1 - \frac{2}{b}x)Q(x)\,dx = -\frac{4}{b\,Q(\tau)}\int_{-b/2}^{\tau-(b/2)} x\,Q\Big(x + (b/2)\Big)\,dx,$$

and this is not zero when $\tau = b$ because Q is positive and increasing on the interval $[0,\infty)$ unless $\gamma = 0$. It *is* possible for the boundary conditions to be satisfied if $\gamma > 0$; for example, $\mathbf{P}^{m+1}$ is a smooth compactification of the total space of $\mathcal{O}_{\mathbf{P}^m}(1)$. $\square$

Proof of Theorem C

The proof of Theorem C differs from the proof of Theorem B in the initial conditions, and consequently in the choice of momentum interval; $\varphi'(0) = 0$ rather than $\varphi'(0) = 2$, so $I = (0,\infty)$ rather than $[0,\infty)$. The details are almost exactly parallel: The profile of equation (3.3) is replaced by

$$\varphi(\tau) = \frac{2}{Q(\tau)}\int_0^\tau (\tau - x)\Big(R(x) - c\Big)Q(x)\,dx, \tag{3.7}$$

which is positive for $\tau > 0$ if $c \ll 0$. An interval $\mathrm{J}^\times$, with supremum equal to $c_0^\times$, is defined as before. The proof that the metrics are complete and live on $\Delta^\times(L)$ if $c < c_0^\times$ is entirely analogous to the proof of Lemma 3.2. The borderline profile is not identically zero if $R(\tau)$ is non-constant, and induces a metric on $L^\times$.

Exactly as before, the metric is Einstein iff

$$\rho_M = \lambda\omega_M \quad\text{and}\quad \frac{1}{2Q}(\varphi Q)'(\tau) = -\lambda\tau, \qquad\text{so}\quad \varphi(\tau) = -\frac{2\lambda}{Q(\tau)}\int_0^\tau xQ(x)\,dx.$$

This function φ is positive on $(0, \infty)$ iff $\lambda < 0$. Conversely, if $\rho_M = \lambda\omega_M$ and $c = \lambda(m+1)$ for some $\lambda < 0$, then an argument analogous to the proof of Lemma 3.4 shows that

$$\Big(R(\tau) - c\Big)Q(\tau) = 2\lambda \frac{d}{d\tau}\Big(-\tau Q(\tau)\Big),$$

so the associated metric is Einstein. □

One significant difference between Theorems B and C appears when R is constant. In the event g_M is Ricci-flat (so $R \equiv 0$) there is no profile satisfying the boundary conditions $\varphi(0) = \varphi'(0) = 0$ that induces a scalar-flat metric. In fact, the equation for scalar-flatness is $(\varphi Q)'' = 0$, while the initial conditions imply $(\varphi Q)(0) = (\varphi Q)'(0) = 0$, so $\varphi \equiv 0$. Even by relaxing the initial conditions, the only scalar-flat metrics that arise are uninteresting. To wit, the equation $(\varphi Q)'' = 0$ leads to the candidate profile $\varphi(\tau) = (a_0 + a_1\tau)/Q(\tau)$. This function induces a complete metric only if it is everywhere positive (and perhaps has a removable discontinuity), but in this event the momentum interval is all of $\mathbf{R}$. The requirement that $\omega_M - \tau\gamma$ be a Kähler form for all $\tau \in \mathbf{R}$ forces $\gamma = 0$, so $Q \equiv 1$ and the profile reduces to a positive constant. The induced metric is a local product of g_M and a flat cylinder of radius equal to the value of the profile.

Bounds on c_0

To facilitate the proof of Lemma 3.3, a rational function C was introduced whose infimum over I is equal to c_0. Since C has an explicit expression in terms of the curvature of the horizontal data, c_0 can in principle be estimated in terms of the curvature. A crude estimate comes from the following simple observations. If $R(\tau) - c \geq 0$ for all $\tau \geq 0$, then the profile (3.3) is positive for $\tau > 0$, while if $R(\infty) - c < 0$, then the profile is *not* always positive. Thus

$$\inf_{\tau \geq 0} R(\tau) \leq c_0 \leq R(\infty). \tag{3.8}$$

To estimate the lower bound, pick a point z of M and choose an orthonormal basis of $T_z^{1,0}M$ relative to which B is diagonal. Denote by β_ν the eigenvalues of B, indexed so that $\beta_1 \leq \cdots \leq \beta_j < 0 = \beta_{j+1} = \cdots \beta_m$, and let ϱ_ν be the diagonal elements of ϱ in the basis. Then

$$R(\tau) = \frac{\varrho_1}{1 - \beta_1\tau} + \cdots + \frac{\varrho_j}{1 - \beta_j\tau} + \varrho_{j+1} + \cdots \varrho_m.$$

Since the coefficients of $R(\tau)$ are by hypothesis independent of z, certain combinations of the ϱ_ν are independent of z. Indeed suppose that among the β_ν the *distinct* real numbers occuring are $b_1 < \cdots < b_{\ell-1} < b_\ell = 0$ with multiplicities $k_1, \dots, k_\ell$, and let

$$r_1 = \varrho_1 + \cdots + \varrho_{k_1}, \quad r_2 = \varrho_{k_1+1} + \cdots + \varrho_{k_1+k_2}, \ \dots, \quad r_\ell = \varrho_{k_\ell+1} + \cdots + \varrho_m.$$

Then $r_\ell = R(\infty)$ and

$$R(\tau) = \sum_{i=1}^{\ell} \frac{r_i}{1 - b_i\tau} = R(\infty) + \sum_{i=1}^{\ell-1} \frac{r_i}{1 - b_i\tau}.$$

Considering each fraction separately, $R(\tau) \geq R(\infty) + \sum_{i=1}^{\ell} \min(0, r_i)$, so that

$$R(\infty) + \sum_{i=1}^{\ell} \min(0, r_i) \leq c_0 \leq R(\infty).$$

Although crude, this is sometimes sufficient to determine c_0 exactly. For example, if ρ_M is positive semi-definite, then all $r_i \geq 0$ and $c_0 = r_\ell = R(\infty) \geq 0$. If in addition γ is negative-definite, then $c_0 = 0$. In either case, the total space of L admits a complete, scalar-flat Kähler metric.

In the setting of Theorem C, there is the additional inequality $c_0^\times \leq R(0)$, necessitated by positivity of the profile (3.7) near $\tau = 0$. If the base metric has non-positive Ricci tensor, then R is monotone increasing and $c_0^\times = R(0) = \sum_\nu \varrho_\nu$.

3.2 Metrics on Vector Bundles

Let $p : (E, h) \to (D, g_D)$ be an Hermitian holomorphic vector bundle of rank $n > 1$ over a Kähler manifold of dimension d. There is a smooth, globally defined norm squared function r, just as for line bundles, and the Calabi ansatz has an obvious formulation in this situation, namely to consider closed $(1,1)$-forms

$$\omega = p^*\omega_D + \sqrt{-1}\partial\bar{\partial}F(r) = p^*\omega_D + 2\sqrt{-1}\partial\bar{\partial}f(t). \tag{3.9}$$

The machinery of Theorem A is easily modified to treat this case. Complex-analytically, the idea is to blow up the zero section of E and pull the form (3.9) back to the total space of the blow-up. The original horizontal data pull back to "partially degenerate" horizontal data since ω_D has rank d as an Hermitian form while the exceptional divisor has dimension $d+n-1$. To avoid problems caused by this degeneracy, the form $\omega_D - \gamma$ on $\mathbf{P}(E^\times)$ is used as a background metric. The following discussion elaborates on the technicalities necessary to make this idea work.

The tautological bundle, and blowing up

The complement of the zero section of E is denoted by $E^\times$, and is regarded either as a complex manifold or as a "punctured $\mathbf{C}^n$-bundle" over D. There is a free $\mathbf{C}^\times$-action on $E^\times$ by scalar multiplication, whose quotient is the *projectivization* of E, denoted $\mathbf{P}(E)$. The complex manifold $M = \mathbf{P}(E)$ is the total space of a $\mathbf{P}^{n-1}$-bundle $\pi : M \to D$. The pullback of E to M is the fibre product

$$\pi^*E = \{(\zeta, v) \in M \times E \mid \pi(\zeta) = p(v)\} = M \times_D E \tag{3.10}$$

endowed with the projection onto the first factor, and the *tautological bundle* τ_E is the line subbundle of $\pi^*E \to M$ whose fibre at a point $\zeta = (z, [v]) \in \mathbf{P}(E) = M$ is the line through $v \in E_z$. Two important observations are:

- The fibre of π over $z \in D$ is the $(n-1)$-dimensional projective space $\mathbf{P}(E_z^\times)$, and the restriction of τ_E to a fibre $\mathbf{P}(E_z^\times)$ is $\mathcal{O}(-1)$. The total space of the latter may be regarded as the blow-up of $\mathbf{C}^n$ at the origin.

- Projection to the second factor in (3.10) induces a biholomorphism $\tau_E^\times \simeq E^\times$, and coincides with the map $\pi : M \to D$ along the zero section of τ_E.

The total space of τ_E is obtained from the total space of E by *blowing up the zero section.* This is a direct generalization of blowing up a point, and indeed may be regarded as a family of blow-ups of $\mathbf{C}^n$, parametrized by points of D.

By abuse of notation, the projection $\tau_E \to M$ is denoted by p. These spaces and bundles are organized as follows, with superscripts denoting ranks of bundles or dimensions of spaces:

$$\begin{array}{ccccc} \pi^*E & \supset & \tau_E & \longrightarrow & E^n \\ \Big\downarrow & & \Big\downarrow p & & \Big\downarrow p \\ \mathbf{P}(E) & = & M^m & \xrightarrow{\pi} & D^d \end{array}$$

Of course, $n + d = m + 1$ since $E^\times \simeq \tau_E^\times$. Via this identification, $L = \tau_E$ acquires an Hermitian structure, also denoted h, from the Hermitian structure of E, and as before $\gamma = \gamma_1(L, h)$ denotes the curvature form. The norm squared function $r : E^\times \to (0, \infty)$ may be regarded as a function on L, and $p^*\gamma = -\sqrt{-1}\partial\bar\partial \log r$ as closed $(1, 1)$-forms on the total space of L. Calabi [4] calls γ the "bi-Hermitian curvature form" of (E, h). The preceeding discussion makes it clear geometrically that the auxiliary bundle L arises naturally in the Calabi ansatz on vector bundles. In spite of the fact that the complex manifolds $L^\times$ and $E^\times$ are biholomorphic, it is best to regard them as differential-geometrically distinct; the bundle $E^\times$ is completed by adding a copy of D (the level set $\{r = 0\} \subset E$), while $L^\times$ is completed by adding a copy of M (the level set $\{r = 0\} \subset L$). Lemma 3.7 below illustrates the importance of this distinction.

Degeneracy in the Calabi ansatz

Let $p : (E, h) \to (D, g_D)$ be an Hermitian holomorphic vector bundle of rank n over a Kähler manifold. The closed $(1, 1)$-form $\pi^*\omega_D$ may be denoted by ω_D for brevity, as in equation (3.11) below. Associated to a profile φ are functions μ and f as in Section 2. Upon blowing up the zero section of E, the closed $(1, 1)$-form $\omega = p^*\omega_D + 2\sqrt{-1}\partial\bar\partial f(t)$ is immediately written, using results of Section 2, as

$$\omega = \varphi(\tau)(dt \wedge d^c t) + p^*\omega_D - \tau p^*\gamma. \tag{3.11}$$

The form $\pi^*\omega_D$ on M is degenerate along the fibres of $\pi : M \to D$, while under the hypotheses of Theorem D the form $\pi^*\omega_D - \gamma$ is a Kähler form on M, and (by Lemma 3.6 below) the horizontal data $p : (\tau_E, h) \to (M, \omega_M)$ are σ-constant. The vector bundle data $p : (E, h) \to (D, \omega_D)$ are said to be σ-constant in this situation. Before defining the analogues of the functions Q and R for vector bundle data, it is reassuring to verify that the choice of background metric in the family $\omega_M(\tau)$ is immaterial.

Lemma 3.6 *If $\omega_M(\tau)$ is a Kähler form for every $\tau > 0$, and if $p : (L, h) \to (M, \omega_M(\tau_0))$ is σ-constant for some $\tau_0 > 0$, then $p : (L, h) \to (M, \omega_M(\tau))$ is σ-constant for every $\tau > 0$.*

Proof Fix $\tau_0 > 0$, and regard $\omega_M(\tau_0)$ as a background metric, so that all index raising and lowering is done with respect to this metric. Let B denote the curvature endomorphism of (L, h);

by assumption, the eigenvalues β_ν are constant on M, and the trace of the Ricci form $\rho_M(\tau_0)$ with respect to the Kähler form $\omega_M(\tau)$ is constant for all $\tau > 0$. Consider the family $A(\tau)$ of endomorphisms associated to the closed $(1,1)$-forms $\omega_M(\tau)$. Then $A(\tau_0)$ is the identity, and $A'(\tau) = -\mathrm{B}$, so

$$A(\tau) = \mathrm{I} - (\tau - \tau_0)\mathrm{B} \qquad \text{for } \tau \geq 0.$$

The eigenvalues of B are $\beta_1, \dots, \beta_d$, and $-1/\tau_0$ (the latter of multiplicity $n-1$). Thus $A(\tau)$ has constant eigenvalues for each $\tau \geq 0$, $A(0) = \mathrm{I} + \tau_0 \mathrm{B}$ has rank d, and the endomorphisms $A(\tau)$ have the same eigenbundles. Set

$$Q(\tau, \tau_0) = \det A(\tau) = \frac{\tau^{n-1}}{\tau_0^{n-1}} \prod_{\nu=1}^{d} \Big(1 - (\tau - \tau_0)\beta_\nu\Big),$$

so that for each $\tau_0 > 0$, Q is a constant-coefficient polynomial with a zero of order $(n-1)$ at $\tau = 0$. Then $Q(\tau, \tau_0)$ is the ratio of the volume forms of $\omega_M(\tau)$ and $\omega_M(\tau_0)$, and is constant on M for each fixed $\tau > 0$, so the Ricci forms $\rho_M(\tau)$ and $\rho_M(\tau_0)$ coincide. Consequently, if the trace of the Ricci form $\rho_M(\tau_0)$ with respect to $\omega_M(\tau)$ is constant for all $\tau > 0$, then the same is true of the Ricci form $\rho_M(\tau_1)$ for every $\tau_1 > 0$. □

Suppose $p : (E, h) \to (D, \omega_D)$ are σ-constant data, and set $\omega_M(\tau) = \pi^*\omega_D - \tau\gamma$. The background metric on M is taken to be $\omega_M := \omega_M(1)$, at variance with the notation used for line bundles. As noted in the proof of Lemma 3.6, the Ricci forms $\rho_M(\tau)$ do not depend on τ, and are denoted simply by ρ_M. By contrast, the Ricci endomorphisms, defined by $\varrho_M(\tau) := \omega_M(\tau)^{-1}\rho_M$, *do* depend on τ, though by hypothesis they all have constant trace. More is true: For each $\tau > 0$, the vertical tangent bundle of M (namely, $\ker \pi_*$) is an eigenbundle of $\varrho_M(\tau)$ with eigenvalue n (of multiplicity $n-1$). To see this, observe that the global inner product $\langle \rho_M, \gamma \rangle$ is constant on M since $\langle \rho_M, \omega_M(\tau) \rangle$ is constant on M for each $\tau > 0$. On a fibre of π, which is a projective space, $-\gamma$ is a Kähler form, and since the trace of ρ_M restricted to the fibre is constant, the restriction must be a multiple of $-\gamma$, which implies the Ricci endomorphism is a multiple of the identity on each fibre. For cohomological reasons, the eigenvalue is n.

The functions Q and R are defined with respect to the background metric ω_M by

$$\begin{aligned} Q(\tau) &= \det A(\tau) = \tau^{n-1} \prod_{\nu=1}^{d} (1 + \beta_\nu - \tau\beta_\nu) =: \tau^{n-1} Q_0(\tau), \\ R(\tau) &= \mathrm{tr}\Big(A(\tau)^{-1}\varrho\Big) = \frac{n(n-1)}{\tau} + \text{smooth}. \end{aligned} \tag{3.12}$$

Note that Q has a zero of order $(n-1)$ and R has a simple pole at $\tau = 0$.

Proof of Theorem D

As in Section 2, a profile φ induces a bundle-adapted metric on E whose scalar curvature is

$$\sigma_\varphi(\tau) = \left(R - \frac{1}{2Q}(\varphi Q)''\right)(\tau). \tag{3.13}$$

The function σ_φ generally has a pole at $\tau = 0$. Interestingly, the scalar curvature is bounded near the zero section if, *and only if*, the metric extends over the zero section:

Lemma 3.7 *If the metric* (3.9) *is complete, then the function* σ_φ *in equation* (3.13) *has a removable singularity at* $\tau = 0$ *if and only if the profile satisfies the boundary conditions*

$$\varphi(0) = 0, \qquad \varphi'(0) = 2,$$

if and only if the metric extends over the zero section of E.

Proof Completeness of the metric dictates that $\varphi(0) = 0$. Writing $\varphi(\tau) = \varphi'(0)\,\tau + O(\tau^2)$ near $\tau = 0$—so that $(\varphi Q)(\tau) = \varphi'(0)\tau^n\, Q_0(\tau) + O(\tau^{n+1})$—and differentiating twice gives

$$\sigma_\varphi(\tau) = \frac{n(n-1)}{\tau}\left(1 - \frac{\varphi'(0)}{2}\right) + O(1),$$

so the singularity at $\tau = 0$ is removable if and only if $\varphi'(0) = 2$. The remaining assertions are proven exactly as in the line bundle case. $\square$

Lemma 3.7 says the Calabi ansatz (3.9) does not give rise to a complete Kähler metric of constant scalar curvature on the punctured disk subbundle of E. Of course, the punctured disk subbundle of E is biholomorphic to the punctured disk subbundle of the line bundle τ_E; the difference between Theorem C and the present situation is that here the forms $\omega_M(\tau)$ drop in rank at $\tau = 0$.

The differential equation obtained by setting (3.13) equal to c has a regular singular point at $\tau = 0$, and the general solution is

$$\varphi(\tau) = \frac{2}{Q(\tau)}\left(\alpha_0 + \alpha_1\,\tau + \int_0^\tau \Big(R(x) - c\Big)(\tau - x)Q(x)\,dx\right).$$

Because Q has a zero of order $n-1$ at $\tau = 0$, $\varphi(0) = 0$ forces $\alpha_0 = \alpha_1 = 0$, so the purported profile for a bundle-adapted metric of scalar curvature c on the disk subbundle of E is

$$\varphi(\tau) = \frac{2}{Q(\tau)}\int_0^\tau \Big(R(x) - c\Big)(\tau - x)Q(x)\,dx. \tag{3.14}$$

This profile satisfies the boundary conditions $\varphi(0) = 0$ and $\varphi'(0) = 2$. The first results from two applications of l'Hôpital's rule. To obtain the latter, differentiate φQ and solve for φ' to get

$$\varphi'(\tau) = -\varphi(\tau)\frac{Q'}{Q}(\tau) + \frac{2}{Q}\int_0^\tau \Big(R(x) - c\Big)Q(x)\,dx.$$

Writing $\varphi(\tau) = \varphi'(0)\tau + O(\tau^2)$ and using l'Hôpital's rule gives

$$\varphi'(0) = \lim_{\tau\to 0}\left(-\varphi'(0)(n-1) + O(\tau) + 2\frac{\Big(R(x) - c\Big)Q(\tau)}{Q'(\tau)}\right) = -\varphi'(0)(n-1) + 2n,$$

so $\varphi'(0) = 2$ as claimed. Most of the remaining points are checked exactly as in the proof of Theorem B. Specifically, the set $\mathrm{J} = \{c \mid \varphi(\tau) > 0 \text{ for all } \tau > 0\}$ is a non-empty interval, bounded above, and $c_0 := \sup \mathrm{J}$. The profile (3.14) is positive for all $\tau > 0$ if $c < c_0$, and has at most quadratic growth, while if $c = c_0$ the induced metric is complete (though possibly of finite fibre volume). When $c > c_0$, there is a first positive root b, and three possibilities occur:

- $\varphi'(b) = 0$, so the metric is complete and of finite volume;
- $\varphi'(b) = -2$, so the metric extends smoothly to $\mathbf{P}(E \oplus \mathcal{O}_D)$, which is the $\mathbf{P}^n$-bundle over D obtained from E by adding a divisor at infinity;
- $\varphi'(b) \neq 0, -2$, so the metric is incomplete and has no smooth extension to a larger manifold.

As in Lemma 3.3, there are at most finitely many values of $c > c_0$ such that one of the first two possibilities holds.

The analysis of the Einstein condition is very similar in this case to the previous two (see Lemma 3.4). One finds that $\rho_\varphi = \lambda\omega_\varphi$ if and only if

$$\lambda\pi^*\omega_D = \rho_M + n\,\gamma \qquad \text{and} \qquad \frac{1}{2Q}(\varphi Q)'(\tau) = n - \lambda\tau.$$

The second equation is a differential equation which leads to the profile whose associated metric has scalar curvature $\lambda(n+d)$. As a consistency check, observe that the form $\rho_M + n\,\gamma$ is cohomologically degenerate along the fibres of π, since ρ_M restricts to the curvature of a Fubini-Study metric, while γ restricts to the negative generator of second cohomology.

Collapsing metrics along the zero section

Theorem D indicates that the momentum construction provides a differential-geometric framework in which to view (partial) collapsing of divisors, provided the normal bundle satisfies curvature hypotheses. The following remarks explain this idea in more detail. Let $p : (E, h) \to (D, g_D)$ be an Hermitian vector bundle satisfying the hypotheses of Theorem D. For each $a > 0$, the horizontal data $p : (L, h) \to M$ equipped with the base metric $\omega_M(a) = \pi^*\omega_D - a\gamma$ are σ-constant, and the "critical" scalar curvature constants $c_0(a)$ are uniformly bounded below for a in some interval to the right of 0. If $c < c_0(a)$ for all such a, then the momentum construction yields a family of metrics on the total space of $L = \tau_E$, each complete and of scalar curvature c. As $a \to 0$, these metrics converge to a metric on the total space of E, i.e. to a closed $(1, 1)$-form on L whose restriction to the zero section has rank d and is otherwise positive-definite. The fibres of $M = \mathbf{P}(E) \to D$ shrink homothetically while the curvature of the fibre metric approaches $-\infty$ along the zero section in such a way that the scalar curvature of the total space stays constant.

4 Limitations, Examples, and Literature Survey

In this section we shall address the following questions:

1. What is the role of the assumption $I = (0, \infty)$ or $[0, \infty)$, made throughout Section 3, and what do the methods say about metrics of finite (fibre) volume, particularly compact metrics?

2. To what extent can the assumption of σ-constancy be relaxed? Are there complete metrics of constant scalar curvature that may be constructed via the Calabi ansatz, but which are not encompassed in the existence theorems of Section 3?

3. How explicitly are the examples described? Are there any "genuinely" new examples?

We shall also give a brief survey of related literature in Section 4.5.

Question 1 is treated in Sections 4.1 and 4.4. As stated, Question 2 is somewhat heuristic, reflecting our lack of a completely definitive answer. Partial results, including the proof of Theorem E, are collected in Section 4.1. Examples of σ-constant data are given in Section 4.2. These provide an answer to Question 3; it emerges that there is greater freedom in choosing the Kähler class of the base metric g_M than there is in prior works. On the question of explicitness, the formulae of Section 3 show that φ is a rational function whose coefficients depend explicitly upon the Ricci endomorphism ϱ and the curvature endomorphism B; see also Section 4.2 below. Section 2 exhibits some important geometric invariants (Ricci/scalar curvature, and Laplace operator) of the metrics very simply and explicitly in momentum coordinates. The price to be paid is that the transformation back to holomorphic coordinates involves inversion of an integral.

4.1 Scope of the Momentum Construction

This subsection describes our attempt to understand the limitations of the Calabi ansatz. The general construction begins with horizontal data—an Hermitian holomorphic line bundle over a complete Kähler manifold, and an interval of real numbers—and associates to each profile φ a Kähler metric g_φ. There are two intervals in the Calabi ansatz: The momentum interval I, which is closely related to the area of the fibre metric; and the r interval defining the invariant subbundle on which the metric is defined. That these intervals are essentially independent is clear from Table 2.2.

Normalization of the momentum interval

Begin with horizontal data $\{p : (L, h) \to (M, g_M),\ I\}$. The horizontal forms $\omega_M - \tau\,\gamma$ are assumed to be positive and complete for all $\tau \in I$, so if $I = \mathbf{R}$, then $\gamma = 0$, and every profile gives rise to a local product metric. In all other cases, I may be normalized without loss of generality; in words, every bundle-adapted metric is isometric to a bundle-adapted metric having momentum interval I equal to $\mathbf{R}$, or else having $\inf I = 0$:

Lemma 4.1 *If $\{p' : (L', h') \to (M, g'_M),\ I'\}$ are horizontal data with $I' \neq \mathbf{R}$ and if ψ is a profile on I' inducing a metric g_ψ, then there exist horizontal data $\{p : (L, h) \to (M, g_M),\ I\}$, with* $\inf I = 0$*, and a profile φ on I such that g_φ and g_ψ are isometric.*

Proof Suppose first that I' is bounded below, and set $a = \inf I'$; thus $\omega'_M - a\gamma'$ is a Kähler form on M. The 'translated' data

$$(L, h) = (L', h'), \qquad \omega_M = \omega'_M - a\gamma', \qquad I = I' - a, \qquad \varphi(\tau) = \psi(\tau - a),$$

have the advertized properties.

Now suppose $I' \neq \mathbf{R}$ is not bounded below. By a translation argument analogous to that just given, it may be assumed that $\sup I' = 0$. Consider the 'inverted' data

$$(L, h) = (L'^{*}, h^{-1}), \qquad \omega_M = \omega'_M, \qquad I = -I', \qquad \varphi(\tau) = \psi(-\tau).$$

The t integral acquires a sign change, which corresponds to the inversion map $\iota : (L, h) \to (L^*, h^{-1})$, given locally by $z^0 \mapsto 1/z^0$ in a line bundle chart. It is straightforward to verify $g_\varphi = \iota^* g_\psi$. $\square$

Level sets of σ_φ

According to Theorem A and equation (2.12), the scalar curvature σ_φ is given in terms of horizontal data and the profile by

$$\sigma_\varphi = \sigma_M(\tau) - \frac{1}{2Q}\frac{\partial^2}{\partial \tau^2}(Q\varphi)(\tau), \qquad \sigma_M(\tau) = R(\tau) - \Box_{\omega_M(\tau)} \log Q; \tag{4.1}$$

Our aim has been to find data (bundle, metrics, interval, and profile) such that σ_φ is constant. As noted in the introduction, it cannot be expected that the level sets of σ_φ—which depend on the curvature of (L, h)—will coincide with the level sets of τ for an arbitrary profile. Similarly, it is not to be expected that the Calabi ansatz yields a metric of constant scalar curvature for arbitrary horizontal data. The goal is thus to determine curvature conditions on the horizontal data that are necessary and sufficient for existence of a profile inducing a metric of constant scalar curvature (or perhaps satisfying some weaker condition, such as having scalar curvature depending only upon τ). The condition of σ-constancy is sufficient, by the existence theorems of the preceeding section. The aim of this subsection (unfortunately not completely realized) is to determine the extent to which σ-constancy is a necessary condition.

Equation (4.1) holds for all profiles and horizontal data. Regard specification of the scalar curvature (4.1) as a family of ODEs on I, parametrized by points of M. This yields a family of solutions, viewed as a function $\varphi : I \times M \to \mathbf{R}$, and the question is to determine when this function depends only on $\tau \in I$. For the moment, assume there is a (smooth) function σ on I so that $\sigma_\varphi = \sigma(\tau)$. Then

$$\varphi(\tau, z) = \frac{1}{Q(\tau, z)}\left[\frac{\partial \varphi}{\partial \tau}(0, z)\,\tau + 2\int_0^\tau (\tau - x)\Big(\sigma_M(x, z) - \sigma(x)\Big)Q(x, z)\,dx\right]. \tag{4.2}$$

Our general goal is to make minimal hypotheses on σ (e.g. σ is real-analytic) and then to prove that if the right-hand side of (4.2) depends only on τ for a particular choice of σ, then the horizontal data are σ-constant.

Proof of Theorem E, and related remarks

Fix horizontal data $\{p : (L, h) \to (M, g_M),\ I\}$. Suppose there exist real-analytic functions σ_1 and σ_2 on I so that the corresponding profiles φ_1 and φ_2 given by (4.2) depend only on τ and agree at $\tau = 0$. Then the function

$$(\varphi_2 - \varphi_1)(\tau) = \frac{1}{Q(\tau, z)}\left[\Big(\varphi_2'(0) - \varphi_1'(0)\Big)\tau + \int_0^\tau (\tau - x)\Big(\sigma_1(x) - \sigma_2(x)\Big)Q(x, z)\,dx\right]$$

depends only on τ and vanishes at 0. Let $N(\tau, z)$ denote the term in square brackets, and let $\psi(\tau) = N(\tau, z)/Q(\tau, z)$. The aim is to show that the horizontal data are σ-constant, namely that $Q(\tau, z)$ depends only on τ and the horizontal metric $g_M(\tau)$ has constant scalar curvature for each $\tau \in I$. Because σ_φ is *assumed* to depend only upon τ (for $\varphi = \varphi_1$, say), it suffices to show that Q depends only on τ, since $\sigma_M(\tau) = \sigma_\varphi + (1/2Q)(\varphi Q)''(\tau)$. To this end, it is enough to show that N depends only on τ since

$$N''(\tau, z) = \Big(\sigma_1(\tau) - \sigma_2(\tau)\Big)Q(\tau, z).$$

The function ψ is real-analytic and not identically zero (unless $\varphi_1 \equiv \varphi_2$). Put $\alpha = \varphi_2'(0) - \varphi_1'(0)$, write $\sigma_1(\tau) - \sigma_2(\tau) = S(\tau)$, and consider the initial-value problem

$$y''(\tau) - \frac{S(\tau)}{\psi(\tau)}y(\tau) = 0, \qquad y(0) = 0, \quad y'(0) = \alpha. \tag{4.3}$$

The idea is that for each $z \in M$, the function $y = N(\ ,z)$ satisfies (4.3), so by uniqueness it follows that N is independent of z. The ODE has a singular point at $\tau = 0$, however, so the desired conclusion requires a modicum of additional work.

If $\sigma_1(\tau) \equiv \sigma_2(\tau)$, then it is clear that $Q(\tau, z)$ does not depend on z, and the proof is finished. Otherwise, $S(\tau) = S_0\tau^\ell + O(\tau^{\ell+1})$ with $S_0 \neq 0$. Since $Q(0, z) = 1$ for all z, $Q(\tau, z)S(\tau) = S_0\tau^\ell + O(\tau^{\ell+1})$, and integrating twice gives

$$N(\tau, z) = \alpha\tau + \frac{S_0}{(\ell+1)(\ell+2)}\tau^{\ell+2} + O(\tau^{\ell+3})$$

near $\tau = 0$. According to whether $\alpha \neq 0$ or $\alpha = 0$, the coefficient S/ψ in (4.3) is given by

$$\tau^{\ell-1}\Big(\frac{S_0}{\alpha} + O(\tau)\Big) \quad \text{or} \quad \frac{(\ell+1)(\ell+2)}{\tau^2}\Big(1 + O(\tau)\Big),$$

cf. (4.4) below, so the ODE has at worst a regular singular point at $\tau = 0$. Fix a point of $z \in M$, and expand N and S/ψ as power series about $\tau = 0$:

$$N(\tau, z) = \sum_{n=0}^\infty a_n(z)\tau^n, \qquad \frac{S(\tau)}{\psi(\tau)} = \frac{1}{\tau^2}\sum_{k=0}^\infty b_k\tau^k.$$

The initial conditions in (4.3) imply $a_0 = 0$ and $a_1 = \alpha$. Setting $y = N(\ ,z)$ in the ODE and equating coefficients gives, suppressing the dependence of a_n on z,

$$a_1 b_0 = 0, \qquad \Big[(n+2)(n+1) - b_0\Big]a_{n+2} = \sum_{k=0}^{n+1} a_k b_{n-k} \quad \text{for} \quad n \geq 0. \tag{4.4}$$

It is convenient to treat the cases $\alpha \neq 0$ and $\alpha = 0$ separately.

$(\alpha \neq 0)$ The function S/ψ has a simple pole at $\tau = 0$, so $b_0 = 0$. By induction on n, the recursion relation (4.4) expresses the coefficients $a_n(z)$ as linear combinations of a_0, a_1, and the b_k. Since the latter are assumed constant, the coefficients $a_n(z)$ do not depend on z, proving that $N(\tau, z)$ depends only on τ.

$(\alpha = 0)$ In this case, N vanishes to order $(\ell + 2)$ at $\tau = 0$, so $a_n(z) \equiv 0$ for $0 \leq n \leq \ell + 1$, while $a_{\ell+2}(z) \equiv S_0/(\ell + 1)(\ell + 2)$ for all $z \in M$; thus $b_0 = (\ell + 2)(\ell + 1)$. As before, the recursion (4.4) expresses the coefficients $a_n(z)$, $n > \ell + 2$, in terms of the constants $a_{\ell+2}$ and b_k, so $N(\tau, z)$ depends only on τ.

This completes the proof Theorem E. $\square$

Even the case $\sigma_1 = \sigma_2 = c$ is of geometric interest: If $\{p,\ I\}$ are horizontal data, and there exist fibre-complete, bundle-adapted metrics of scalar curvature c in $\Delta(L)$ (so that $\varphi_1'(0) = 2$) and $\Delta^\times(L)$ (i.e. $\varphi_2'(0) = 0$), then the data are σ-constant, cf. Theorems B and C. It is worth emphasizing that Theorem E is local; nothing is assumed about completeness of the base or fibre metric. Though the result is geometrically significant, the proof is elementary because of Theorem A and the assumption of existence of *two* profiles whose scalar curvature is independent of τ; taking the difference $\varphi_2 - \varphi_1$ cancels the relatively complicated (but profile-independent) term $\sigma_M(\tau)$.

By Theorem E, it is reasonable to assert that the Calabi ansatz is exhausted, so far as families of Kähler metrics of constant scalar curvature are concerned. More precisely, if $\{p,\ I\}$ is *not* σ-constant, then there is at most one metric of constant scalar curvature that arises from the Calabi ansatz in the disk bundle $\Delta(L)$. For the remainder of this section, we consider the more difficult question of deducing curvature properties from existence of a *single* profile.

Question 4.2 Let $\{p,\ I\}$ be horizontal data, and suppose there exists a profile φ depending only on τ such that σ_φ is constant. Are the horizontal data necessarily σ-constant?

We are not presently able to answer this question, though we believe the answer is "yes." While falling short of a proof, our calculations provide evidence and yield some suggestive partial results.

The approach is to draw conclusions about the curvature of horizontal data under the assumption that there exists a germ of a profile at $\tau = 0$, satisfying $\varphi(0) = 0$ and $\varphi'(0) = 2$, and inducing a bundle-adapted metric whose scalar curvature depends only on τ. This investigation only involves consideration of rational functions defined on $I \times U$, with U a neighbourhood of the origin in $\mathbf{C}^m$. Solutions of ODEs defining momentum profiles are considered in the 'non-geometric regime' where $\tau < 0$. Nothing is being asserted about bundle-adapted Kähler metrics for $\tau < 0$, of course; what is being used is real-analyticity of φ in τ, and the fact that if such a function $\varphi(\tau, z)$ depends only on τ for $\tau \geq 0$, then the same is true for $\tau < 0$.

Proposition 4.3 *Fix horizontal data $\{p : (L, h) \to (M, g_M),\ I\}$ and a polynomial function σ. If there exists a profile $\varphi(\tau)$ such that $\sigma_\varphi = \sigma(\tau)$, then φ is a* rational *function.*

Proof Fix $z \in M$, and let $\varphi(\tau, z)$ be the function defined by (4.2). It will be shown that a non-rational term in φ has a logarithmic singularity at a root of Q, and that this root must depend

on z, so φ depends on z as well. These logarithmic terms potentially arise from $\Box_{\omega_M(\tau)} \log Q$, which enters via $\sigma_M(\tau) = R(\tau) - \Box_{\omega_M(\tau)} \log Q$.

It is enough to work in a coordinate neighbourhood U around $z \in M$. Let $\{\beta_\nu(z)\}_{\nu=1}^m$ denote the (*a priori* non-constant) eigenvalues of the curvature endomorphism B in U, and let $b_1 < \cdots < b_\ell \le 0$ be the *distinct* eigenvalues, of multiplicity k_i; without loss of generality, k_i may be assumed constant throughout U. Then

$$Q(\tau, z) = \prod_{\nu=1}^m \Big(1 - \tau\beta_\nu(z)\Big) = \prod_{i=1}^\ell \Big(1 - \tau b_i(z)\Big)^{k_i}, \qquad \log Q(\tau, z) = \sum_{i=1}^\ell k_i \log\Big(1 - \tau b_i(z)\Big).$$

Choose a unitary eigenframe $\{\mathbf{e}_\nu\}_{\nu=1}^m$ in U, and let (z^ν) be complex local coordinates such that for $1 \le \nu \le m$, $z^\nu = 0$ and $\partial_\nu = \mathbf{e}_\nu$ at z. Differentiating $\log Q(\tau, z)$, using subscripts (after a comma) to denote partial derivatives, gives

$$\partial_\lambda \bar\partial_\mu \log Q = \sum_{i=1}^\ell \left(\frac{k_i \tau^2 b_{i,\lambda} b_{i,\bar\mu}}{(1 - \tau b_i)^2} - \frac{k_i \tau b_{i,\lambda\bar\mu}}{(1 - \tau b_i)} \right). \tag{4.5}$$

Let $\{\theta_\nu\}$ be the basis of $\bigwedge_{\mathbf{0}}^{(1,0)}$ dual to $\{\mathbf{e}_\nu\}$, so along the fibre at $\mathbf{0}$ the metric and its inverse are given by

$$\omega_M(\tau) = \sum_{\nu=1}^m (1 - \beta_\nu \tau)\, \theta_\nu \bar\theta_\nu, \qquad \omega_M(\tau)^{-1} = \sum_{\nu=1}^m \frac{e_\nu \bar e_\nu}{(1 - \beta_\nu \tau)}.$$

Taking the trace of (4.5) with respect to $\omega_M(\tau)^{-1}$, and using the fact that $\{\mathbf{e}_\nu\}$ is a coordinate basis at $\mathbf{0}$,

$$\Box_{\omega_M(\tau)} \log Q(\tau, z) = \sum_{i=1}^\ell \sum_{\lambda=1}^m \left(\frac{k_i \tau^2 |b_{i,\lambda}|^2}{(1 - \tau\beta_\lambda)(1 - \tau b_i)^2} - \frac{k_i \tau b_{i,\lambda\lambda}}{(1 - \tau\beta_\lambda)(1 - \tau b_i)} \right) \tag{4.6}$$

at $\mathbf{0}$. When the right-hand side is expanded in partial fractions, the coefficients of the principal part at $\tau = 1/b_i$ are linear combinations of derivatives of b_i. In particular, if a principal part appears, then b_i is non-constant.

The integrand in (4.2) is rational for each $z \in M$, and the only non-polynomial terms come from (4.6). In other words, $\varphi(\tau, z)$ is the integral of a rational function, and any non-rational (i.e. logarithmic) terms arise from a principal part. But as just observed, a principal part is non-vanishing only when some b_i is non-constant, and this implies φ depends on z. $\square$

Under similar hypotheses, the calculations in the proof can be used to read off other geometrically interesting consequences. When the curvature endomorphism B is a multiple of the identity, examination of the highest-order pole in (4.6) implies σ-constancy:

Proposition 4.4 *Let $\{p,\ I\}$ be horizontal data such that $\mathrm{B}(z) = \beta(z)\mathrm{I}$ for some smooth function β. Assume there is a profile $\varphi(\tau)$ inducing a metric whose scalar curvature is $\sigma(\tau)$ for some polynomial σ. Then the data are σ-constant.*

Finally, observe that the non-constant curvature eigenvalues are tightly constrained by the assumptions made above.

Proposition 4.5 *Let $\{p,\ I\}$ be horizontal data, and let $\{b_i(z)\}_{i=1}^{\ell}$ denote the distinct eigenvalues of the curvature endomorphism. Suppose there exists a profile, depending only on τ and inducing a metric whose scalar curvature depends only on τ. If $b_i(z)$ is non-constant, then for every $z \in M$, $(1-\tau b_i)^{k_i}$ divides the term in square brackets in* (4.2).

Proof By hypothesis, φ—and in particular each pole—is independent of z. So if $(1-\tau b_i)^{k_i}$ does not divide the numerator (the term in square brackets) in (4.2) then b_i is independent of z. $\square$

The condition that the numerator should have such a factor is a strong constraint on the way in which $b_i(z)$ can vary. Unfortunately, it is a constraint whose consequences seem hard to express in simple fashion.

4.2 Line Bundles

This section describes some examples of data satisfying the hypotheses of Theorems B and C. The "atomic" examples are suitable line bundles over Hodge manifolds of constant scalar curvature; even when the base is a curve, interesting (and seemingly new) metrics are obtained.

Scalar-flat metrics on complex surfaces

It is instructive to see what emerges from the momentum construction when the (complex) dimension of the base M is 1. (The case $\dim M = 0$ was dealt with in Section 2.3, see Table 2.2.)

The conditions of σ-constancy are satisfied iff $\rho_M = \lambda\omega_M$ and $\gamma = \beta\omega_M$, for constants λ and β. Then $Q(\tau) = 1-\beta\tau$ and $R(\tau) = \lambda(1-\beta\tau)^{-1}$. The profiles with $\varphi(0) = 0$ which yield a metric of constant scalar curvature c are given, with their derivative at 0, by

$$\varphi(\tau) = \frac{2\tau + (\lambda - c)\tau^2 + c\beta\tau^3/3}{1-\beta\tau}, \qquad \varphi'(0) = 2; \tag{4.7}$$

or

$$\varphi(\tau) = \frac{(\lambda - c)\tau^2 + c\beta\tau^3/3}{1-\beta\tau}, \qquad \varphi'(0) = 0. \tag{4.8}$$

It is easy to analyze the choices of β, λ, and c that give rise to positive profiles and hence to complete metrics. However, the most interesting case is that where $c = 0$, for then the metric is anti-self-dual in the sense of 4-dimensional conformal geometry. In addition to recovering a number of known examples, we find new metrics on $\mathbf{C}^2$ and its quotients by $\mathbf{Z}$ and $\mathbf{Z} \oplus \mathbf{Z}$.

The first case to consider is when $\lambda > 0$, so $M = \mathbf{P}^1$ with a Fubini-Study (round) metric, normalized so that $\lambda = 1$. In this case, if $L = \mathcal{O}(-k)$, then $\beta = -k/2$; the factor of 2 arises because the canonical bundle is $\mathcal{O}(-2)$. Equation (4.7) takes the form

$$\varphi_k(\tau) = \frac{2\tau + \tau^2}{1 + k\tau/2},$$

which is clearly positive on $(0,\infty)$ if $k \geq 0$. The growth is linear at ∞, so the corresponding metric ω_k lives on the line bundle (not some disk subbundle).

By Theorem B, ω_k is Einstein iff $k = 2$, in which case it is an example of the Ricci-flat metric on $T^*\mathbf{P}^d$ found by Calabi [4], also known as the Eguchi-Hanson graviton. If $k = 1$ the metric is the Burns metric on the blow-up of $\mathbf{C}^2$ at the origin [18, 19]. If $k \geq 3$ the metrics are those found by LeBrun in [18].

In the case $\lambda < 0$, equation (4.7) never yields complete metrics; the corresponding profiles are all negative for large τ, but if b is the first positive zero then $\varphi'(b) = -2$ iff $\beta = 0$. This case does, however, yield compact extremal Kähler metrics, see [35].

Finally, if $\lambda = 0$—so the universal cover of M is $\mathbf{C}$—then (4.7) reduces to

$$\varphi(\tau) = \frac{2\tau}{1 - \beta\tau}. \tag{4.9}$$

This profile yields a complete metric on $L = \mathcal{O}_{\mathbf{C}}$, whose total space is $\mathbf{C}^2$, provided $\beta < 0$. These complete, scalar-flat Kähler metrics on $\mathbf{C}^2$ are not Einstein, and appear to be new. The formula $h_\beta := \exp(\beta(\operatorname{Im} z)^2)$ defines an Hermitian metric of constant, negative curvature; indeed, $\gamma(\mathcal{O}_{\mathbf{C}}, h_\beta) = \beta\omega$, where $\omega = (\sqrt{-1}/2)dz \wedge d\bar{z}$ is the standard Kähler form.

If the Hermitian metric h is suitably translation-invariant, then the profile (4.9) gives rise to complete metrics of zero scalar curvature on $(\mathbf{C}/\mathbf{Z}) \times \mathbf{C}$ and on the total space of a line bundle of negative degree over an elliptic curve. If β is integral, h_β is an example of such a $\mathbf{Z} \oplus \mathbf{Z}$-invariant metric.

Bundles over product manifolds

The basic example of σ-constant horizontal data comes from a combination of some well-known results:

Lemma 4.6 *Let (M, g) be a Hodge manifold of constant scalar curvature. Then there exists an Hermitian line bundle $p : (L, h) \to (M, g)$ such that the data $\{p, [0,\infty)\}$ are σ-constant.*

Proof By the Lefschetz Theorem on $(1,1)$-classes, there exists a holomorphic line bundle L whose first Chern class is $[-\omega]$. To see there is an Hermitian structure h with $\gamma(L, h) = -2\pi\omega$, start with an arbitrary Hermitian structure h_0 and let γ_0 be the curvature form. By the Hodge Theorem, there exists a unique smooth, real-valued function u satisfying

$$\gamma - \gamma_0 = \sqrt{-1}\partial\bar{\partial}u, \qquad \int_M u \,\mathrm{dvol}_g = 0.$$

Put $h = e^{-u}h_0$. For this Hermitian structure, the horizontal forms $\omega_M(\tau)$ are positive—indeed, are homothetic to ω_M—hence have constant scalar curvature for all $\tau \geq 0$. Finally, the curvature endomorphism is a scalar multiple of the identity, in particular has constant eigenvalues. □

There are trivial improvements on the statement; the Kähler form need only be *homothetic* to an integral form, and every positive power of L admits a suitable Hermitian structure. It is also

clear that there are examples of σ-constant horizontal data over certain non-compact manifolds, such as Hermitian symmetric spaces.

Remark 4.7 It is tempting to 'iterate' the momentum construction, using a constant scalar curvature metric g_φ on a disk bundle $\Delta(L)$ as the base metric for appropriate horizontal data. While this is sometimes possible, it is noteworthy that the 'natural' choice of line bundle—L pulled back over its own total space, equipped with the induced Hermitian structure—does *not* fit into this framework. Indeed, if (L, h) is non-flat, then the curvature of the pullback bundle, computed with respect to an *arbitrary* bundle-adapted metric on the disk bundle, does not have constant eigenvalues. □

For data as in Lemma 4.6, the functions P, Q, and R are written explicitly as follows: Let the constant scalar curvature of the Hodge manifold (M^m, g) be σ, and let $p : (L, h) \to (M, g)$ be the Hermitian line bundle with curvature $\gamma = -2\pi\omega$. For each positive integer k, the bundle (L^k, h^k) has curvature $k\gamma$. Fix $\alpha > 0$, and equip M with the metric $g_M = 2\pi\alpha\, g$. Together with the compatible interval $[0, \infty)$, these data are σ-constant, and

$$Q(\tau) = \left(1 - \frac{k}{\alpha}\tau\right)^m, \qquad R(\tau) = \frac{\alpha\sigma}{\alpha - k\tau}, \qquad P(\tau) = 2\sigma\left(1 - \frac{k}{\alpha}\tau\right)^{m-1}. \tag{4.10}$$

Since every positive or negative Einstein-Kähler metric is (homothetic to) a Hodge metric, Lemma 4.6 encompasses most prior examples of horizontal data to which Theorems B and C may be applied. It is reasonable to ask whether or not a Hodge metric of constant scalar curvature is "essentially" Einstein-Kähler. The answer is "no," even discounting product metrics, homogeneous metrics, and the like. LeBrun [20] constructed scalar-flat metrics on certain blown-up ruled surfaces; the Kähler classes of these metrics depend on a real parameter, and the class is rational when the parameter value is rational. Higher-dimensional examples include the manifolds obtained from $\mathbf{P}^{2k+1}$ by blowing up a pair of skew $\mathbf{P}^k$'s. Such a manifold is a $\mathbf{P}^1$-bundle over $\mathbf{P}^k \times \mathbf{P}^k$, and a Kähler class is determined by the area of a $\mathbf{P}^1$ fibre and by the areas of lines in each of the base factors. If the latter are equal, and are a rational multiple of the area of a fibre, then the Kähler class is proportional to a Hodge class, and by [12] (Remark 5.3, p. 584) is represented by a metric of constant scalar curvature.

The family $g_M(\tau)$ of horizontal metrics arising from σ-constant horizontal data need not be homothetic. A simple way to arrange this is to take a suitable product of data arising from Lemma 4.6:

Lemma 4.8 *Let $\{p_j : (L_j, h_j) \to (M_j, g_j),\ [0, \infty)\}$, $j = 1, \dots, n$, be σ-constant horizontal data. Then the line bundle*

$$p : L = \bigotimes_{j=1}^{n} \pi_j^* L_j \longrightarrow M = M_1 \times \cdots \times M_n, \qquad \pi_j : M \to M_j \quad \text{the projection,}$$

equipped with the induced metrics, is σ-constant.

The proof is immediate, and is left to the reader. The corresponding function Q is a product of terms as in equation (4.10), while R is a sum of such terms.

If (M_j, g_j) are Einstein-Kähler, then the construction just described gives rise to complete metrics on tensor products of pluri-canonical and pluri-anticanonical bundles as in Corollaries B.2 and B.3.

Remarks about compact Einstein-Kähler metrics

By Yau and Aubin's solution of the Calabi conjecture, simply-connected Einstein-Kähler manifolds of non-positive scalar curvature are plentiful, the simplest examples being smooth complete intersection varieties of degree $k \geq N+1$ and dimension $m \geq 2$ in the complex projective space $\mathbf{P}^N$.

Work of Tian and Yau [34] and Tian [33] shows that, with precisely two exceptions, every compact complex *surface* with positive first Chern class admits an Einstein-Kähler metric.

Remark 4.9 (Deformation of positive Einstein-Kähler manifolds) In higher dimensions, work of Nadel [25] and Siu [31] shows that if $m/2 \leq k \leq m+1$, then the Fermat hypersurface of degree k in $\mathbf{P}^{m+1}$ admits an Einstein-Kähler metric with positive Ricci curvature. Nadel also proved existence of positive Einstein-Kähler metrics on certain branched coverings of projective space. These examples can be deformed; this is an immediate consequence of a Theorem of LeBrun and Simanca [21] regarding deformations of extremal Kähler metrics. Precisely, if (M, J, g_M) is a positive Einstein-Kähler manifold with no non-trivial holomorphic vector fields, their result implies that every sufficiently small deformation of the complex manifold (M, J) also admits an Einstein-Kähler metric. $\square$

If (M, J) is a smooth hypersurface of degree $k \leq m+1$ in $\mathbf{P}^{m+1}$, then every small deformation is realized by a hypersurface in the same projective space. Consequently, there is a moduli space of dimension $\binom{m+k+1}{k} - (m+2)^2 + 1$ (the dimension of the space of monomials minus the dimension of $\operatorname{Aut}\mathbf{P}^{m+1}$) consisting of Einstein-Kähler structures (complex structure and compatible Kähler metric) on the smooth manifold underlying the Fermat hypersurface. Since for $m \geq 3$ every smooth, irreducible hypersurface in $\mathbf{P}^{m+1}$ has $h^{1,1} = b^2 = 1$ (by the Lefschetz Hyperplane Theorem), every holomorphic line bundle over such a manifold admits an Hermitian structure for which the line bundle data are σ-constant.

4.3 Vector Bundles

This section describes some examples of data satisfying the hypotheses of Theorem D. As mentioned in the introduction, σ-constancy for a vector bundle of rank $n > 1$ is a strong condition, nowhere nearly as flexible as the corresponding notion for line bundles. This is expected, since the base metric g_D lives on a space of dimension d, while it is necessary to control $d + n - 1$ curvature eigenvalues. The examples here are all chosen so that the exceptional divisor of the blow-up is particularly simple, either a homogeneous space, or else a product $D \times \mathbf{P}^{n-1}$. Nonetheless, there are interesting metrics, many of which seem to be new.

Homogeneous vector bundles

Let D be a *compact*, homogeneous Kählerian manifold, and fix a maximal compact group $K \subset \operatorname{Aut}^0(D)$, endowed with a bi-invariant measure of unit volume. The group K is unique up to

conjugacy in $\mathrm{Aut}^0(D)$. Furthermore, K acts transitively on D, and every de Rham class (in particular, every Kähler class) contains a *unique* K-invariant representative, obtained from an arbitrary representative by averaging.

Remark 4.10 When D is simply-connected, it is known that D is rational, and that every holomorphic vector bundle over D is homogeneous. Generally, a compact, homogeneous Kähler manifold is the Kähler product of a flat torus and a rational homogeneous space, see Matsushima [24], or Besse [3] for a more detailed expository treatment. □

The Ricci form of a K-invariant Kähler metric is K-invariant and represents $2\pi c_1(D)$. In other words, there is only one K-invariant Ricci form, and its eigenvalues, with respect to a K-invariant Kähler form are K-invariant functions, i.e. constants.

Suppose $p : E \to D$ is a homogeneous holomorphic vector bundle, i.e. is induced by a representation of K on $GL(n, \mathbf{C})$, and that the ruled manifold $M = \mathbf{P}(E)$ is K-homogeneous (which happens, e.g., when E is irreducible). Then the tautological bundle $p : L \to M$ is a homogeneous line bundle, and by averaging over K it is clear that for every Kähler class on M, there exists a Kähler form ω_M representing the chosen class, and an Hermitian structure in L, such that the data $p : (L, h) \to (M, g_M)$ are σ-constant.

A tubular neighbourhood of the zero section of L is obtained by blowing up a tubular neighbourhood of the zero section of E, so it is clear that these neighbourhoods are either both pseudoconvex or both not pseudoconvex. But in L, pseudoconvexity is equivalent to non-positivity of the curvature. Thus, under the assumptions of Corollary D.1, the hypotheses of Theorem D are satisfied, so the total space of E (or the disk subbundle) admits complete metrics of constant scalar curvature.

Finally, consider the K-invariant Ricci form ρ_M on M. The restriction of ρ_M to a fibre of $\pi : M \to D$ is the Ricci form of a Fubini-Study metric, so the form $\rho_M - n\gamma$ vanishes on $\ker \pi_*$, hence is pulled back from D. Provided this form is non-positive, there is an Einstein-Kähler metric on the total space of E or the disk subbundle. We shall not pursue this avenue; a detailed (partial) classification of cohomogeneity-one Einstein-Kähler metrics has been accomplished by Dancer and Wang [6], see also Podesta and Spiro [28].

Sums of line bundles

Let $E = \Lambda \otimes \mathbf{C}^n$ be a sum of n copies of an Hermitian line bundle over a base space such that $(\Lambda, h) \to (D, g_D)$ is σ-constant. Then the projectivization is a product manifold $M = D \times \mathbf{P}^{n-1}$, and the tautological bundle is tensor product $\tau_E = \pi_1^*\Lambda \otimes \pi_2^*\mathcal{O}_{\mathbf{P}^{n-1}}(-1)$. It is clear that the induced Hermitian structure on $L = \tau_E$ is σ-constant with respect to the obvious metric on M.

As a partial complement to this remark, observe that if L_1 and L_2 are holomorphic line bundles over a *compact* manifold D, and if $L_1^* \otimes L_2$ admits a non-trivial holomorphic section, then there do not exist metrics g and h such that the data $(L_1 \oplus L_2, h) \to (D, g)$ are σ-constant. This generalizes the remark about Hirzebruch surfaces that was made following Theorem D. As in that case, the projective bundle $M = \mathbf{P}(L_1 \oplus L_2)$ has non-reductive automorphism group, hence (by a theorem of Lichnerowicz) cannot admit a Kähler metric of constant scalar curvature. Considerations of this type rather seriously restrict the possibility of finding sums of line bundles to which Theorem D applies. For example, if $D = \mathbf{P}^d$ (or more generally, is a compact, irreducible, rank-one Hermitian

symmetric space), then the *only* sums of line bundles satisfying the hypotheses of Theorem D are as in the previous paragraph.

Stable bundles over curves

Many of the observations below regarding ruled manifolds over curves were made by Fujiki in the context of seeking extremal Kähler metrics on ruled manifolds, see [9].

Let C be a smooth, compact Riemann surface of genus at least two, and let $E \to C$ be a holomorphic vector bundle of rank n and degree k. We will use a theorem of Narasimhan-Seshadri [26], as formulated by Atiyah-Bott [1], to show that when E is stable there is an Hermitian metric in E satisfying the hypotheses of Theorem D.

Equip C with the (unique up to isometry) Kähler form ω_C of unit area and constant Gaussian curvature, so that $c_1(E) = k\omega_C$. In [1, §§6 and 8] it is explained that stable holomorphic bundles E correspond to irreducible representations $\Gamma_R \to U(n)$, where Γ_R is a central extension of the fundamental group:

$$0 \to \mathbf{R} \to \Gamma_R \to \pi_1(C) \to 0.$$

It follows that if E is stable then there is a family of metrics with constant Ricci eigenvalues on $\mathbf{P}(E)$. Indeed, because the universal cover Δ of C is Stein (and contractable) the universal cover of $\mathbf{P}(E)$ may be identified holomorphically with $\Delta \times \mathbf{P}(\mathbf{C}^n)$. Equipping this with a product metric Poincaré $\times$ Fubini-Study, it follows that if E is stable then $M = \mathbf{P}(E)$ is the quotient $\Delta \times \mathbf{P}(\mathbf{C}^n)/\pi_1(C)$ where the action is by *isometries*. Scaling the two factors gives a two-parameter family of Kähler forms on M whose Ricci eigenvalues are constant, and whose eigenbundles are the vertical tangent bundle of M and its orthogonal complement. Thus $h^{1,1}(M) \geq 2$.

The Leray-Hirsch Theorem (see, e.g., [14], pp. 31ff.) implies that, as a module over $H^*(C, \mathbf{R})$, the cohomology ring $H^*(M, \mathbf{R})$ is generated by the first Chern class ζ of the tautological bundle τ_E subject to the relation

$$0 = \zeta^n - c_1(E)\,\zeta^{n-1} = \zeta^n - k\omega_C\,\zeta^{n-1}. \tag{4.11}$$

In particular, $h^2(M) = 2$, so by the observations made in the previous paragraph, every two-dimensional cohomology class is represented by a form whose pullback to $\Delta \times \mathbf{P}^{n-1}$ is a (possibly indefinite) linear combination of the Poincaré form and the Fubini-Study form. Kähler classes are exactly those classes whose representatives pull back to positive combinations of these metrics. Fix a Kähler class on M, and let g_M be the distinguished representative. Every holomorphic line bundle $p : L \to (M, g_M)$ admits an Hermitian structure h, unique up to scaling, whose curvature form is a combination of the Poincaré and Fubini-Study forms; thus the data $p : (L, h) \to (M, g_M)$ are σ-constant.

Lemma 4.11 *Let $E \to C$ be a holomorphic vector bundle of rank n and degree k over a compact Riemann surface of genus $g \geq 2$. Assume C and $M = \mathbf{P}(E)$ are equipped with metrics as above, and let ω_F denote the push-forward to M of the integral Fubini-Study form on $\Delta \times \mathbf{P}^{n-1}$. Then the curvature form γ of the tautological bundle of E and the Ricci form ρ_M are given by*

$$\frac{1}{2\pi}\gamma = \frac{k}{n}\omega_C - \omega_F, \qquad \frac{1}{2\pi}\rho_M = (2 - 2g)\,\omega_C + n\,\omega_F. \tag{4.12}$$

Proof Write $\gamma = k_1\,\omega_C + k_2\,\omega_F$. Then $k_2 = -1$ since the restriction of τ_E to a fibre is $\mathcal{O}_{\mathbf{P}^{n-1}}(-1)$. By equation (4.11) and a short calculation, $k_1 = k/n$. To see the Ricci form is as claimed, pull back to the universal cover, where the metric is a product, and recall that ω_C has unit area. $\square$

If $L \to M$ is a line bundle whose first Chern class is non-positive, then the hypotheses of Theorems B and C are satisfied. Further, by Lemma 4.11 the tautological bundle $L = \tau_E$ has non-positive first Chern class iff $k = \deg E \leq 0$. In this event, the hypotheses of Theorem D are satisfied, and the total space of E (or the disk subbundle) admits complete Kähler metrics of constant scalar curvature. Finally, Lemma 4.11 implies $\rho_M + n\gamma = 2\pi(2g-2+k)\,\omega_C$, so if $k = 2-2g$, then by Theorem D the total space of E admits a complete, Ricci-flat Kähler metric, while if $k < 2-2g$, the disk subbundle of E supports a complete Einstein-Kähler metric of negative curvature.

4.4 Metrics of Finite Fibre Area

Metrics of finite fibre area arise in the momentum construction when the momentum interval is bounded. In this case the most convenient normalization is to take the momentum interval I to be symmetric about $\tau = 0$, rather than insisting that $\inf I = 0$. The requirement that $\omega_M(\tau) = \omega_M - \tau\gamma$ be positive-definite for all $\tau \in I$ no longer imposes a sign restriction on γ. The main difficulty is that the problem of constructing a complete, finite-volume Kähler metric of constant scalar curvature is overdetermined: There are two boundary conditions—namely $\varphi = 0$, and $|\varphi'| = 0$ or 2—that must be satisfied at each end of the momentum interval, and only three parameters (the initial conditions, and the value of the scalar curvature). Consequently, it is to be expected that the Kähler class of g_M will be restricted by existence of a complete metric with constant scalar curvature.

Metrics on compact manifolds

Theorem 4.12 below is an existence result for Kähler metrics of constant (positive) scalar curvature on certain compact manifolds to which the momentum construction is applicable. It was proven in [12], and is included here for two purposes: To suggest the type of theorem to be expected for non-compact (but finite-volume) metrics, and to indicate the parts of the proof that generalize with no extra effort. The important philosophical point is that on a compact manifold, it is not generally the case that every Kähler class is represented by a metric of constant scalar curvature.

Theorem 4.12 *Let (M, g_M) be a product of positive Einstein-Kähler manifolds, each having $b_2 = 1$, and let $p : L \to M$ be a holomorphic line bundle whose first Chern class is strictly indefinite. Then the completion $\widehat{L} = \mathbf{P}(L \oplus \mathbf{1})$ admits a Kähler metric of constant scalar curvature. In fact, the set of Kähler classes containing such a metric is a real-algebraic family that separates the Kähler cone.*

Proof (Sketch) The first step is to establish that under the hypotheses of Theorem 4.12, every Kähler class on the compact manifold $\widehat{L}$ is represented by an extremal Kähler metric (in the sense of Calabi). For present purposes, this may be taken to mean that the scalar curvature is an affine function of the momentum coordinate τ, i.e. that the gradient field of the scalar curvature is a global holomorphic vector field. The terminology comes from a variational problem having the

latter property as its Euler-Lagrange equation, see [5]. A Kähler class containing an extremal representative is an *extremal class.*

Let Q and R be defined as in Section 2, and let $I = [-b, b]$ be the momentum interval. Proceeding backward, set $\sigma(\tau) = \sigma_0 + \sigma_1\tau$ and attempt to solve the boundary value problem

$$(\varphi Q)''(\tau) = 2Q(\tau)\Big(R(\tau) - \sigma_0 - \sigma_1\tau\Big); \qquad \varphi(\pm b) = 0, \quad \varphi'(\pm b) = \mp 2. \tag{4.13}$$

The values of σ_0 and σ_1 are determined uniquely by Q, R, and b (Lemma 4.13 below), and an elementary (but slightly involved) root counting argument shows that $(\varphi Q)''$ vanishes at most twice, so that φQ—which is positive near the endpoints of I—is positive on $(-b, b)$. Thus equation (4.13) determines a momentum profile whose induced metric is extremal, proving that every Kähler class on $\widehat{L}$ contains an extremal representative (possibly with non-constant scalar curvature). The root counting argument rests crucially on the fact that the curvature of the base metric is non-negative.

The proof is completed by determining conditions under which $\sigma_1 = 0$, see also (4.16) below. This is accomplished by expressing σ_1 as a polynomial in the curvature and Ricci eigenvalues of the horizontal data, so that the "variables" are exactly the parameters controlling the Kähler class of the base metric. Roughly, the top coefficient changes sign as the parameters vary, so σ_1 changes sign on the Kähler cone, hence vanishes on a real-algebraic hypersurface that separates the cone. The corresponding metrics have constant scalar curvature. □

There are substantial difficulties in extending Theorem 4.12 to compact manifolds when the base curvature is not positive. Perhaps the greatest, found by Tønnesen-Friedman [35], is that the set of extremal classes is not obviously the entire Kähler cone. In particular, on a ruled surface whose base has genus at least two, the set of classes for which the momentum construction yields an extremal metric is *not* generally the entire Kähler cone. This potentially complicates the final portion of the argument, since σ_1 may vanish for certain choices of eigenvalues, but the relevant parameters may not correspond to extremal classes.

Non-compact metrics of finite fibre area

By analogy with the compact case, it is desirable to search among a family of metrics whose scalar curvature may be non-constant. The natural extension is to the class of *formally extremal* metrics, by definition those whose scalar curvature is an affine function of the moment map. As before, the hope is to find, for each pair of boundary conditions, an affine function $\sigma_0 + \sigma_1\tau$ such that the function φ satisfying

$$R(\tau) - \frac{1}{2Q(\tau)}(\varphi Q)''(\tau) = \sigma_0 + \sigma_1\tau \tag{4.14}$$

matches the given boundary conditions and is non-negative. Matching the boundary conditions is easy linear algebra:

Lemma 4.13 *Let $\{p : (L, h) \to (M, g_M), [-b, b]\}$ be σ-constant horizontal data for some $b > 0$. For each pair of boundary values $\varphi'(\pm b)$, there exists a unique choice of σ_0 and σ_1 such that the function φ defined by* (4.14) *has the given boundary derivatives and satisfies $\varphi(\pm b) = 0$.*

Proof For $n \geq 0$, set

$$\alpha_{[n]} = \int_{-b}^{b} x^n Q(x)\, dx, \qquad A_{[n]} = -\tau^n (\varphi Q)'(\tau)\Big|_{-b}^{b} + \int_{-b}^{b} x^n (QR)(x)\, dx.$$

Observe that $\alpha_{[n]}/\alpha_{[0]}$ is the nth moment (in the sense of probability) of the moment map τ, computed with respect to the symplectic measure, and that $\alpha_{[2]}\alpha_{[0]} - \alpha_{[1]}^2 > 0$ by the Cauchy-Schwarz Inequality. Integrating (4.14), using the given boundary conditions at $\tau = -b$, yields

$$(4.15)\qquad \begin{aligned} (\varphi Q)'(\tau) &= (\varphi Q)'(-b) + 2\int_{-b}^{\tau} \Big(R(x) - \sigma_0 - \sigma_1 x\Big) Q(x)\, dx, \\ (\varphi Q)(\tau) &= (\varphi Q)'(-b)(\tau + b) + 2\int_{-b}^{\tau} (\tau - x)\Big(R(x) - \sigma_0 - \sigma_1 x\Big) Q(x)\, dx. \end{aligned}$$

Setting $\tau = b$ and subtracting b times the first equation from the second leads to the system

$$\begin{bmatrix} \alpha_{[0]} & \alpha_{[1]} \\ \alpha_{[1]} & \alpha_{[2]} \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \end{bmatrix} = \begin{bmatrix} A_{[0]} \\ A_{[1]} \end{bmatrix}.$$

This system has a unique solution since the coefficient matrix is non-singular. $\square$

This "boundary-matching" result is the only part of the proof of Theorem 4.12 that generalizes immediately, and the results of Tønnesen-Friedman suggest that there are genuine geometric complications in less restricted settings. It is appropriate to remark here that

$$(4.16)\qquad \sigma_1 = 0 \qquad \text{iff} \qquad \alpha_{[0]}A_{[1]} - \alpha_{[1]}A_{[0]} = 0.$$

In the compact case, this is exactly the condition that the Futaki character of a Kähler class vanishes; in the present situation, it is to be expected that there is an invariant for "finite volume Kähler classes" on certain non-compact manifolds. However, it is not immediately obvious whether or not the profile is positive on a neighbourhood of $\tau = \pm b$, regardless of whether or not $\sigma_1 = 0$; in the compact case, this is automatic because the boundary derivatives are non-zero.

4.5 Other Constructions of Bundle-Adapted Metrics

Several authors have constructed bundle-adapted metrics from various points of view. With the exception of LeBrun, who worked over curves, the authors mentioned below have used the following curvature hypotheses, either implicitly or explicitly:

(i) The eigenvalues of the curvature endomorphism B are constant on M;

(ii) The eigenvalues of the Ricci endomorphism ϱ are constant on M;

(iii) At each point of M, B and ϱ are simultaneously diagonalizable.

For brevity, data satisfying these conditions are said to be *ρ-constant*. Data which are ρ-constant are clearly σ-constant. Roughly, the distinction is between assuming an endomorphism has constant trace and assuming its eigenvalues are constant.

Historical survey

The list below is in approximate chronological order, but does not necessarily follow lines of development back to their earliest discernible origins.

Calabi [4] used distortion potential functions to construct complete Einstein-Kähler metrics in line bundles over an Einstein-Kähler base, and in the cotangent bundle of $\mathbf{P}^d$, and used the same method (in [5]) to construct compact "extremal" Kähler metrics of non-constant scalar curvature.

Koiso and Sakane [16, 17] used the momentum map as a coordinate to construct compact Einstein-Kähler metrics of real cohomogeneity one. Their work followed Sakane [29], likely inspired by the work of Bérard-Bergery [2] on compact, non-homogeneous Einstein metrics. One salient point of their work is an explicit interpretation of vanishing of the Futaki invariant as an integrability condition for an ODE. At about the same time, Mabuchi [22] gave a more symplectic proof of existence of Einstein-Kähler metrics on the same spaces considered by Koiso and Sakane. Mabuchi also gave a satisfactory interpretation of the Futaki invariant in terms of convex geometry.

A symplectic approach was also taken by LeBrun [19], and his formalism was generalized by Pedersen and Poon [27]. LeBrun assumed that the base M was a curve, but allowed the profile to be a general (positive) function on $I \times M$. (Correspondingly the dependence upon τ of $\omega_M(\tau)$ is not necessarily affine, and $d\omega_\varphi = 0$ is an additional condition.) He showed that *every* S^1-invariant scalar-flat Kähler metric in complex dimension 2 is locally described by a pair of functions u and w on $I \times M$ satisfying certain partial differential equations. His w is essentially the reciprocal of our φ. What is remarkable is that these equations are tractable when $\dim M = 1$ and lead, for example, to the construction of scalar-flat Kähler metrics on (certain) blow-ups of ruled surfaces.

Pedersen and Poon allowed $\dim M > 1$ and worked with torus bundles, but made assumptions that reduce their very complicated system of equations to the ODEs studied in this paper. (Indeed in most of their examples, M is Einstein–Kähler and L is a (possibly fractional) power of the canonical bundle.) Their examples include metrics of constant scalar curvature on line and disk bundles over projective spaces. In particular they showed that there exist Kähler metrics of zero scalar curvature on the total space of $\mathcal{O}(-m) \to \mathbf{P}_n$ provided that $m \geq n$. In fact this restriction on m is unnecessary, as was shown by Simanca [30], also using the Calabi ansatz.

The methods of Koiso and Sakane were used in [12], and independently by Guan [10], to extend Calabi's families of extremal metrics. Hwang ([12], p. 564) attributed the construction to Koiso and Sakane, overlooking the fact that Calabi ([4], p. 281, equation (4.9), for example) had written the moment map and distance function in the manner of equation (2.2) above. However, Calabi seems to have made the observation in passing, and did not emphasize the use of momentum coordinates.

Koiso [15] and Guan [11] used the method of Koiso and Sakane to study Hamilton's Ricci flow for Kähler metrics and how it may fail to converge; the concept of a *quasi-Einstein* Kähler metric is introduced in the latter two papers. A quasi-Einstein Kähler metric is the Kählerian analogue of a *Ricci soliton*, introduced by R. Hamilton.

Dancer and Wang [6] and Podesta and Spiro [28] have independently used the techniques of Koiso and Sakane to obtain a partial classification of Einstein-Kähler metrics having real hypersurface orbits under the action of the isometry group.

Tønnesen-Friedman [35] used the Calabi ansatz to study existence of extremal Kähler metrics on some ruled surfaces. Some of her examples are of great interest for the following reason. If

(M, J) is a compact complex manifold that admits an extremal Kähler metric in some Kähler class, then there are two properties that might generally be hoped for:

- *Every* Kähler class contains an extremal representative;
- Extremal metrics in a fixed class on M are unique up to the action of the automorphism group.

Tønnesen-Friedman found families of complex surfaces (admitting extremal metrics) for which at least one of the preceeding statements fails.

The papers of Calabi [4], Koiso-Sakane [16], and Pedersen-Poon [27] develop versions of the momentum construction from the points of view of Kählerian, Riemannian, and symplectic geometry, respectively. The remainder of this section gives short summaries of each method (in the notation of the present paper where applicable), including dictionaries between these constructions and the momentum construction as presented here.

The Calabi construction: Kähler distortion potentials

Let $p : (E, h) \to (D, g_D)$ be an Hermitian holomorphic vector bundle of rank n over a Kähler manifold of dimension d. In a coordinate chart $V \subset D$ over which E is trivial, there exist holomorphic coordinates $z = (z^\alpha)$, $\alpha = 1, \dots, d$ for D and fibre coordinates—i.e. local holomorphic sections of E—denoted $\zeta = (\zeta^i)$, $i = 1, \dots, n$, which together constitute a chart for the total space of E. The Hermitian structure is given in this chart by an Hermitian matrix-valued function $H = (H_{i\bar{\jmath}}) : V \to \mathbf{C}^{n\times n}$, and the norm squared function r is given by $r = \zeta^i \bar{\zeta}^j\, H_{i\bar{\jmath}}(z)$. The canonical Hermitian connection is given locally by the form $\Theta = H^{-1}\partial H$, and the curvature form is $\Omega = \bar{\partial}\Theta$.

Let $E' \subset E$ be an invariant (disk) subbundle, in the sense of Definition 1.2. For every strictly subharmonic function $\widehat{F} : E' \to \mathbf{R}$, $\omega = p^*\omega_D + \sqrt{-1}\partial\bar{\partial}\widehat{F}$ is a Kähler form on E'. Following Calabi [5], $\widehat{F}$ is called a *distortion potential* for ω. The idea is to construct a Kähler form on E' by lifting the "horizontally supported" form ω_D and adding a form which is non-degenerate in the fibre directions. Some care is required since $\sqrt{-1}\partial\bar{\partial}\widehat{F}$ has horizontal components, of course.

The Calabi ansatz is to choose $\widehat{F} = F(r)$. Thus

$$\omega = p^*\omega_D + \sqrt{-1}\partial\bar{\partial}F(r). \tag{4.17}$$

Explicitly, the assumption is that the level sets of $\widehat{F}$ coincide with the level sets of r. A short calculation gives

$$\omega = \Big(F'(r) + rF''(r)\Big)\left[\sqrt{-1}\frac{\partial r}{r} \wedge \frac{\bar{\partial} r}{r}\right] + p^*\omega_D - \Big(rF'(r)\Big)\,\gamma,$$

where the form γ is the "bi-Hermitian curvature form" of (E, h), namely, the curvature form of the tautological line bundle $L \to \mathbf{P}(E)$, pulled back to E.

Calabi [4] searches for Einstein-Kähler metrics, i.e. Kähler metrics with $\rho = k\omega$, whose (locally defined) Kähler potential function Φ satisfies the complex Monge-Ampère equation

$$\det(\Phi_{\alpha\bar{\beta}}) = |\mathrm{hol}|^2\, e^{-k\Phi}$$

with |hol| the absolute value of a non-vanishing (local) holomorphic function on D. Assuming further that g_D is Einstein-Kähler and (E, h) is a line bundle with constant curvature, Calabi reduces existence of an Einstein-Kähler metric to an ODE, and solves this equation in terms of a polynomial function which, in our notation, is φQ. Finally, he gives criteria (in terms of the horizontal data) for completeness of the resulting metric, as in Theorem B above. Similar methods are applied to the cotangent bundle of $\mathbf{P}^d$.

Under Calabi's curvature hypotheses, the scalar curvature of (4.17) is constant on the level sets of r, and is therefore specified by an ordinary differential expression in F. Unfortunately, this expression is fourth-order and fully nonlinear, so it is not trivial (or always possible) to write the metrics explicitly or to understand their geometry. In the case of a single fibre (i.e. a bundle over a point), the scalar curvature of the Kähler form (4.17) is

$$\sigma = -e^{-\psi(r)}\Big(\psi'(r) + r\psi''(r)\Big), \qquad \psi(r) = \log\Big(F'(r) + rF''(r)\Big);$$

generally, curvature terms from the bundle (E, h) and base metric (D, g_D) enter. Solving the equation $\sigma = c$ for F is sometimes possible, see Simanca [30], who treats line bundles over a complex projective space.

The Koiso-Sakane ansatz

The approach of Koiso and Sakane [16] is more in the spirit of Riemannian geometry than Kähler geometry. Divide the $\mathbf{C}^\times$-bundle $L^\times$ by the natural circle action, obtaining $(0, \infty) \times M$, then seek a function $s : (0, \infty) \to \mathbf{R}$ and a family $\{g_s\}$ of Riemannian metrics on M such that forming the warped product metric $ds^2 + g_s$ on $(0, \infty) \times M$, lifting back to $L^\times$, and taking the Hermitian metric "$ds^2 + (ds \circ J)^2 + g_s$" yields a Kähler metric. They determine that the family g_s must be of the form $g_M - \tau B$ (a special case of the Duistermaat-Heckman theorem), and change coordinates so that the moment map τ is the "independent variable."

As we have emphasized, the geometry of ω is more easily extracted from the momentum description than from the distortion potential description due to the interplay between φ and τ. To reiterate, the "independent variable" t depends only on the complex structure of the line bundle L. By contrast, the "independent variable" $\tau = \mu(t)$ *depends on the choice of profile*; the coordinate in which the metric is described is unknown data at the outset. It is remarkable that the non-linearity of the scalar curvature—which is locked into the description when using a fixed holomorphic coordinate system—is absorbed into the unknown momentum map, leaving only a "universal" second-order *linear* operator. Holomorphic coordinates do not always provide the most transparent geometric description, even for Kähler metrics.

The Hamiltonian constructions of LeBrun, Pedersen and Poon

Holomorphic coordinates are also suppressed in this approach. Start with a complex manifold M, an interval I and a circle bundle $L' \to I \times M$, equipped with a $U(1)$-connection. The aim is to give L' an S^1-invariant Riemannian metric that is Kähler with respect to a complex structure to be constructed from data on $I \times M$. Let Θ be the connection 1-form on L', w a positive function

on $I \times M$, and τ an affine parameter on I. The holomorphic structure of L' is specified by taking $\mathrm{w}\, d\tau + i\Theta$, along with the $(1,0)$ forms pulled back from M, to be of type $(1,0)$. The ansatz for the Kähler form is

$$\omega = d\tau \wedge \Theta + \omega',$$

where ω' is a family of $(1,1)$-forms on M, parametrized by I. It follows from these definitions that $1/\mathrm{w}$ is the length-squared of X, the generator of the $U(1)$-action on L'. In general the conditions of integrability and closure lead to a complicated system of equations; these can be found in [27], where the generalization to torus bundles was also considered.

Things are simpler when $\dim M = 1$, the case considered by LeBrun [19]. He wrote $z = x + iy$ for a complex coordinate on M and put

$$\omega = d\tau \wedge \Theta + \mathrm{w}e^u dx \wedge dy$$

for some smooth u on $I \times M$. The conditions that ω be closed and the almost-complex structure integrable are given by the PDE

$$\mathrm{w}_{xx} + \mathrm{w}_{yy} + (\mathrm{w}e^u)_{\tau\tau} = 0. \tag{4.18}$$

The condition $\sigma = 0$ is equivalent to the so-called $SU(\infty)$ Toda lattice equation

$$u_{xx} + u_{yy} + (e^u)_{\tau\tau} = 0. \tag{4.19}$$

The metric arises from the Calabi ansatz when w is independent of x and y, since $\mathrm{w}^{-1} = \varphi(\tau)$. In this event, equation (4.18) forces $\mathrm{w}e^u$ to be an affine function of τ, precisely in accord with the momentum construction.

When ω is highly symmetrical it can happen that there is a choice of circle action that yields a description in terms of the momentum construction, while another choice of circle action does not. For example, the scalar-flat 'Burns metric' ω on $\widetilde{\mathbf{C}}^2$, the blow-up of $\mathbf{C}^2$ at the origin, arises in the momentum construction from the identification $\widetilde{\mathbf{C}}^2 = \mathcal{O}(-1) \to \mathbf{P}^1$. However, the Burns metric is $U(2)$-invariant, and may be symplectically reduced with respect to the S^1-action given in standard coordinates on $\mathbf{C}^2$ by $(Z_1, Z_2) \mapsto (e^{i\theta}Z_1, Z_2)$. LeBrun calculated the functions w and u, and found $e^u = 2\tau$ but a complicated expression for w, not independent of (x, y). In particular, with this choice of horizontal data ω does *not* arise from the momentum construction!

References

[1] M. F. Atiyah and R. Bott: *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Royal Soc. London A, **308** (1982), 524–615.

[2] L. Bérard-Bergery: *Sur de nouvelles variétés Riemannian d'Einstein*, Pub. de l'Institute E. Cartan, Nancy, **4** (1982), 1–60.

[3] A. L. Besse: Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete; 3 Folge, Band 10, Springer-Verlag, Berlin, New York, 1987.

[4] E. Calabi: *Métriques Kähleriennes et fibrés holomorphes*, Annales Scientifiques de l'École Normale Supérieure **12** (1979), 268–294.

[5] E. Calabi: *Extremal Kähler metrics*, in Seminars on differential geometry (S. T. Yau, ed.), Annals of Math. Studies, Princeton Univ. Press, 1982, 259–290.

[6] A. Dancer and M. Y. Wang: *Kähler-Einstein metrics of cohomogeneity one, and a bundle construction for Einstein-Hermitian metrics*, preprint, 1997.

[7] M. Engman: *New spectral characterization theorems for S^2*, Pacific J. Math., **154** (1992), 215–229.

[8] M. Engman: *Trace formulæ for S^1 invariant Green's operators on S^2*, Manuscripta Math., **93** (1997), 357–368.

[9] A. Fujiki: *Remarks on extremal Kähler metrics on ruled manifolds*, Nagoya Math. J., **126** (1992), 89–101.

[10] Z.-D. Guan: *Existence of extremal metrics on compact almost homogeneous Kähler manifolds with two ends*, Trans. AMS **347** (1995), no. 6, 2255–2262.

[11] Z.-D. Guan: *Quasi-Einstein metrics*, Int. J. Math **6** (1995) no. 3, 371–379.

[12] A. D. Hwang: *On existence of Kähler metrics with constant scalar curvature*, Osaka J. Math., **31** (1994), 561–595.

[13] S. Kobayashi: Transformation groups in differential geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete; Band 70, Springer-Verlag, Berlin, New York, 1972.

[14] S. Kobayashi: Differential geometry of complex vector bundles, Publications Nihon Sugakki **15**, Iwanami Shoten (Tokyo), Princeton University Press (Princeton), 1987.

[15] N. Koiso: *On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics*, in Recent topics in differential and analytic geometry, Advanced Studies in Pure Math **18-I**, 1990, 327–337.

[16] N. Koiso and Y. Sakane: *Non-homogeneous Kähler-Einstein metrics on compact complex manifolds* in Curvature and topology of Riemannian manifolds, Springer Lecture Notes in Mathematics **1201**, 1986, 165–179.

[17] N. Koiso and Y. Sakane: *Non-homogeneous Kähler-Einstein metrics on compact complex manifolds II*, Osaka J. Math., **25** (1988), 933–959.

[18] C. Lebrun: *Counter-Examples to the Generalized Positive Action Conjecture*, Commun. Math. Phys. **118** (1988) 591–596.

[19] C. Lebrun: *Explicit self-dual metrics on $\mathbf{C}P_2\sharp\ldots\mathbf{C}P_2$*, J. Diff. Geom., **34** (1991) 223–253.

[20] C. Lebrun: *Scalar-flat Kähler metrics on blown-up ruled surfaces*, J. reine angew. Math., **420** (1991), 161–177.

[21] C. Lebrun and S. R. Simanca: *Extremal Kähler metrics and complex deformation theory*, Geometric and Functional Analysis, **4** (1994), 298–336.

[22] T. Mabuchi: *Einstein-Kähler forms, Futaki invariants, and convex geometry on toric Fano varieties*, Osaka J. Math, **24** (1987), 705–737.

[23] Y. Matsushima: *Remarks on Kähler-Einstein manifolds*, Nagoya Math. J., **46** (1972), 161–173.

[24] Y. Matsushima: *Sur les espaces homogènes kählériens d'un groupe de Lie réductif*, Nagoya Math. J., **11** (1957), 53–60.

[25] A. M. Nadel: *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Annals of Math., **46** (1990), 549–596.

[26] M. S. Narasimhan and C. S. Seshadri: *Stable and unitary vector bundles on compact Riemann surfaces*, Annals of Math. **82** (1965), 540–567.

[27] H. Pedersen and Y.-S. Poon: *Hamiltonian construction of Kähler-Einstein metrics and Kähler metrics of constant scalar curvature*, Comm. Math. Phys. **136** (1991), 309–326.

[28] F. Podesta and A. Spiro: *Kähler manifolds with large isometry group*, preprint, 1997.

[29] Y. Sakane: *Examples of Einstein-Kähler manifolds with positive Ricci tensor*, Osaka J. Math. **23** (1986), 585–616.

[30] S. R. Simanca: *Kähler metrics of constant scalar curvature on bundles over* $\mathbf{CP}_{n-1}$, Math. Ann. **291** (1991), 239–246.

[31] Y.-T. Siu: *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical bundle and suitable finite symmetry group*, Annals of Math. **127** (1988), 585–627.

[32] I. A. Taimanov: *Surfaces of revolution in terms of solitons*, Ann. Global Analysis and Geom., **15**, No. 5, Oct. 1997, 419–435.

[33] G. Tian: *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. **101** (1990), 101–172.

[34] G. Tian and S.-T. Yau: *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*, in Mathematical Aspects of String Theory, San Diego, California, S.-T. Yau ed., World Scientific, 1986, pp. 574–628.

[35] C. Tønnesen-Friedman: *Extremal Kähler metrics on ruled surfaces*, Odense Univ. Preprint No. 36, 1997.